Lectures on the ABC conjecture over function fields

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1 Introduction.

One of the main objectives in mathematics is solving algebraic equations. Given a system of algebraic equations $G(X) = 0$ defined over a ring A, may we know if there are solutions of it on A? and, in case, may we explicitly find them? Of course the prototype of such a ring is $\mathbb{Z}$ or rings finite over it, but the theorem of Matiyasevich tells us that we cannot find a general method to answer to the question in this situation.

Never the less one can try to see if it is possible to answer to the question for some class of systems of algebraic equations.

Since long time we know that there is an interesting analogy between the ring $\mathbb{Z}$ and the ring $k[t]$ where k is a field:

– They are both principal ideal domains with Krull dimension one.

– In both rings a product formula holds: Over $\mathbb{Z}$ the product of the all the possible absolute values of an element is one; over $k[t]$ every rational functions has as many poles as many zeroes on the projective line.

Essentially all the technics one can develop to study the system of polynomial equations over $\mathbb{Z}$ may be developed also for studying the same theory over $k[t]$. But on the ring $k[t]$ we can use a new tool: we can compute the derivative of a polynomial. Thus one can hope that the study of the diophantine equations may be easier to solve over $k[t]$ then over $\mathbb{Z}$. Consequently before we attack the theory of systems of polynomial equations over $\mathbb{Z}$, we can try to study the same theory over $k[t]$. In most of the cases, a statement which is false in the theory of polynomial equations over $k[t]$ is also false (often for the same reasons) in the theory over $\mathbb{Z}$.

Of course over $k[t]$ we can use all the strength of the algebraic and analytic geometries.

In these notes we will overview some of the known results and technics used in the theory of the polynomial equations over $k[t]$. Essentially nothing is original, up perhaps the mistakes, the inaccuracies and the choice of the presentation.

Instead of studying systems of polynomial equations over $k[t]$ we will begin by studying the same problem over its field of fractions $F := k(t)$. Given a system of polynomial equations $G(X) = 0$, say in n variables over F, we can see the set of its solution as an affine variety X. As it is customary in algebraic geometry, we can compactify this affine variety to obtain a projective variety $\overline{X}$. Moreover, since $G(X)$ has its coefficients in F, we may consider it as a system of polynomial equations in the variables (t, X) and look its set of zeroes as an affine variety $\mathcal{X}$ defined over k. The variety $\mathcal{X}$ is equipped with a rational map to the affine line $\mathbb{A}^1_k$ given by the projection $(t, X) \to t$. We will see that the morphism $\mathcal{X} \to \mathbb{A}^1_k$ and its compactifications will be a main tool in the subject.

In the first part of these notes we will try to translate the problem we are interested in, in terms of algebraic geometry. We will see, how the derivative will play an interesting role. Then we will introduce the theory of heights: this is a machinery which is very useful and it is essentially the classical intersection theory reinterpreted in the language of arithmetic geometry.

Once we developed the language and the tools, we will introduce the main conjecture: this is a conjecture due to Vojta and, a proof of it will allow to qualitatively solve all the systems of polynomial equations. In the simpler case it is the classical ABC theorem proved by Mason.

Using the analogy between the number fields arithmetic and the function fields arithmetic we can translate the ABC conjecture in the arithmetic contest. In this case the ABC conjecture is widely open even in the simplest case. Let's state it:

1.1 Conjecture. *Let $\epsilon > 0$, then there exists a constant $C(\epsilon)$ for which the following holds: Let a, b and c three integral numbers such that $(a,b) = 1$ and $a + b = c$ then*

$$\max\{|a|, |b|, |c|\} \leq C(\epsilon) \left(\prod_{p/abc} p \right)^{1+\epsilon}$$

where the product is taken over all the prime numbers dividing abc.

Once one state the conjecture in an algebro geometric language and generalize it to an arbitrary curve it becomes:

1.2 Conjecture. *Let $\epsilon > 0$, and K be a number field, $\pi : X \to \mathrm{Spec}(\mathcal{O}_K)$ a regular arithmetic surface and $D \hookrightarrow X$ an effective divisor on X. Denote by $K_{X/\mathcal{O}_K}$ the relative dualizing sheaf. Then there exists a constant $C := C(X, \epsilon, D)$ for which the following holds: let L be a finite extension of K and $P : \mathrm{Spec}(\mathcal{O}_L) \to X$, not contained in D, then*

$$h_{K_{X/\mathcal{O}_K}(D)}(P) \leq (1+\epsilon)(N^{(1)}_D(P) + \log|\Delta_L|) + C[L : K]$$

where $h_{K_{X/\mathcal{O}_K}(D)}(P)$ is the height of P with respect to $K_{X/\mathcal{O}_K}(D)$ and Δ_L is the relative discriminant.

One should compare this conjecture with the conjecture 6.2.

We will provide some different approaches to the ABC conjecture 6.2 in the function fields case. Approaches developed essentially by Vojta, Moriwaki, Kim, McQuillan and others. We will concentrate our attention to the case of curves. In this case we will see that analytic tools will essentially completely solve the conjecture. Since one would like to translate the methods in the arithmetic context (over $\mathbb{Z}$) we would be interested in a purely algebraic proof of it. Thus we will see what we can obtain in this case and, and we will see some counterexamples in positive characteristic. The existence of these counterexamples shows that it is impossible to find an algebraic proof of the ABC conjecture which is independent on the characteristic.

We tried to keep these notes as self contained as possible. We suppose that the reader is acquainted with a standard text in algebraic geometry (for instance [Ha]) Unfortunately we did not completely succeed on this: the last sections require a little bit more of background. We tried nevertheless to recall the properties of the objects we use.

2 Rational points, models and integral points.

Let k be an algebraically closed field and B be a smooth projective curve over it. In the sequel we will denote by F the field $k(B)$.

Unless it is expressively declared, we will suppose from now on that k is of characteristic zero.

The field B is a *non algebraically closed field*. We would like to study the geometry and the arithmetic of varieties defined over it.

Let $f : X_F \to S := \mathrm{Spec}(F)$ be a variety defined over F.

If L/F is a finite extension, we will denote by $X_F(L)$ the set of L–rational points of X_F.

2.1 *What is a rational point?.* In arithmetic geometry we are interested on study the set $X_F(F)$ of F rational points of X_F. More generally, suppose that L/F is a finite extension, we are interested on the study of $X_F(L)$ of L rational point of X_F.

First of all we would like to understand more geometrically what is a rational point.

2.1 Example. Let $F = k(t)$ and X_F be the line $\{X + tY + tZ = 0\} \subset \mathbb{P}^2$. Then the point $[0 : 1 : -1]$ is a rational point of X_F. But also $[-t; 1; 0]$ is a rational point of X_F.

In the example above we see that the two points on the variety X_F are of different nature: the first one has coordinates in the small field k and the second had coordinates over the field F. Also notice that, in order to define the points and the curve, we used coordinates (thus an embedding inside a projective plane). We want give a geometric definition of rational point, which do not depend on the coordinates and that is intrinsic.

We make another example:

2.2 Example. Let k be *any* field. Let X be the variety $\mathbb{A}^1_k := \mathrm{Spec}(k[t])$. A k rational point on X is simply an element $a \in k$. It correspond to the maximal ideal $(t-a) \subset k[t]$. It also corresponds to a k–morphism $k[t] \to k$. And it also corresponds to a k–morphism $a : \mathrm{Spec}(k) \to X$.

In general we see that the example above generalize to any affine variety: If A is a k algebra and $X = \mathrm{Spec}(A)$; Suppose that $X \subset \mathbb{A}^N_k$. A point $(a_1, \ldots, a_N)$ of $\mathbb{A}^N_k$ with coordinates in k which is contained in X corresponds to a maximal ideal $m_a \subset A$ such that $A/m_a \simeq k$; thus:

There is a bijection between k–rational points of X and morphisms of k-schemes $\mathrm{Spec}(k) \to X$.

From this observation, it is natural to give the following definition:

2.3 Definition. *Let F be a field and $f : X_F \to \mathrm{Spec}(F)$ be a F–variety. A F–rational point of X_F is a morphism of F–Schemes*

$$P : \mathrm{Spec}(F) \longrightarrow X_F.$$

The set of F–rational points of X_F is denoted $X_F(F)$.

2.4 Remark. Observe that, if L/K is a field extension and X_F is a variety defined over F, denoting X_L the L–variety $X_F \times_F \mathrm{Spec}(L)$, we have that

$$X_F(F) \subseteq X_L(L).$$

(prove this by exercise).

2.6 *Models of varieties.* At the beginning of this section we used an intuitive argument using specified values of parameters. In this subsection we would like to formalize this. A variety X over, for instance, the field $F = k(t)$, is defined by some equations where the coefficients are elements of F. If we consider the constant t as a variable, the equations defining X define a variety over k. This variety will be a *model of* X.

2.6 Example. Consider the X curve over field $k(t)$ defined by the equation $y^2 = x^3 + t^2$. We may consider t as a variable and associate to X the *surface* $\mathcal{X}$ defined over k defined by the same equation (the variables will be x, y *and* t). The surface $\mathcal{X}$ has a k–morphism $f : \mathcal{X} \to \mathbb{A}^1$; sending $(x; y; t)$ to t. Each time we fix a point $t_0 \in \mathbb{A}^1$ we may look to the fibre of f over t_0; it will be the k–curve $X_{t_0} := \{y^2 = x^3 + t_0^2\}$. Observe that the curve X_{t_0} is smooth if $t_0 \neq 0$ and singular if $t_0 = 0$. Observe also that the surface $\mathcal{X}$ is singular exactly where the curve X_0 is singular. If we blow up the singular point and then we blow up once again the singular point of the strict transform of $\mathcal{X}$ be obtain a smooth surface $\tilde{\mathcal{X}}$. This is *another model of* X. Observe that this model has again a map over $\mathbb{A}^1$ but this time the fibre over 0 is a reducible curve.

On the other side if you consider the curve $Y^2 = X^3 + t$, we the model we associate to it is a smooth surface but the fibre over 0 remains singular.

We fix our function field in one variable F over k. Let X_F be a variety over F. A model of X_F over k will be a k variety which generalize the example above. We will see that the models are not unique and we will study the relations between them. As the example above shows, there are models which are better then others (for instance they are smooth k–varieties if X_F is smooth).

Recall that F is the field $k(B)$ where B is a smooth projective curve over k. The field F is equipped with a map of k–schemes $\eta : \mathrm{Spec}(F) \to B$. The image is a point everywhere dense called *the generic point of* B.

2.7 Definition. *Let $f : X_F \to \mathrm{Spec}(F)$ be a variety. A* model *of X_F over B is a k–variety $\mathcal{X}$ equipped with a flat surjective map of k varieties $g : \mathcal{X} \to B$ and such that $X_F = \mathcal{X} \times_B \mathrm{Spec}(F)$; in other words the following diagram is cartesian:*

$$\begin{array}{ccc} X_F & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathrm{Spec}(F) & \xrightarrow{\eta} & B \end{array}$$

2.8 *Main properties of the models.*

(a) by definition, X_F is the fibre over the generic point of B. Since η is dense in B, we have that the image of X_F in $\mathcal{X}$ is dense.

(b) Let $F(X_B)$ be the fraction field of X_F, then $F(X_F) = k(\mathcal{X})$ (prove it by exercise)

(c) Let $p \in B$ be a closed point and $Y \subset \mathcal{X}$ be a subvariety contained in the fibre over p; Let $\tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up of Y; then $\tilde{X}$ is another model of X_F.

(d) More generally prove the following fact:

2.8 Proposition. *Let $\mathcal{X}$ be a k variety with an isomorphism of fields $F(X_F) \simeq k(\mathcal{X})$ then, there is a variety $\tilde{\mathcal{X}}$ birational to $\mathcal{X}$ with a flat surjective map $\tilde{\mathcal{X}} \to B$ which is a model of X_F.*

(e) A natural model for the projective space $\mathbb{P}^n_F$ is the product $B \times_k \mathbb{P}^n_k$.

(f) Suppose that X_F is projective, an easy (but not always good) way to construct a model of X_F is the following:

Embed $X_F \hookrightarrow \mathbb{P}^N_F$. Consider the model of $\mathbb{P}^N_F$ constructed in (e) and take the Zariski closure of X_F inside it. The Zariski closure $\mathcal{X} \hookrightarrow B \times_k \mathbb{P}^N_k$ is then a model of X_F.

(g) Other natural models of $\mathbb{P}^N_F$ are constructed as follows: Let E be a vector bundle of rank $N+1$ over B, then $\mathbb{P}(E) \to B$ is a smooth projective model of $\mathbb{P}^N_F$; of course we can apply (f) to this situation and construct in this way other models of projective X_F.

(h) Suppose that R is a k–scheme such that $k(R) = F$. Typical examples R are affine open sets of B or the spectra of local rings of closed points of B. As before we have a natural map $\eta : \mathrm{Spec}(F) \to R$. We can generalize the notion of model to R: It will be a faithfully flat R–scheme $\mathcal{X}_R \to R$ such that $X_F \simeq \mathcal{X}_R \times_R \mathrm{Spec}(F)$. In this case we will say that $\mathcal{X}_R$ is a model of $\mathcal{X}_F$ over R.

(i) Suppose that $\mathcal{X}$ is a model of X_F over, say B. Let $p_0 : \mathrm{Spec}(k) \to B$ be a closed point. Then we may consider the k–variety $X_{P_0} := \mathcal{X} \times_B p_0$. This construction is the formal version of the argument of "specializing the parameters" used at the beginning of this section.

(j) Suppose that $h : B' \to B$ is a finite covering of curves. Let F' be the field of functions of B' (it is a finite extension of F). Let $\mathcal{X} \to B$ be a model of X_F. Let $X_{F'}$ be the F'–variety $X_F \times_F \mathrm{Spec}(F')$. Then $\mathcal{X}' =: \mathcal{X} \times_B B'$ is a model of $X_{F'}$ over F'. Warning: even if $\mathcal{X}$ is a k–smooth variety, in general $\mathcal{X}'$ will not be a smooth variety.

(k) By the theorem of resolution of singularities (We suppose to be in characteristic zero), every projective variety X_F over F has a model which is a smooth k–variety.

(l) Suppose that X_F is smooth. From now on we will always suppose that the models of it are normal. Observe that every model is dominated by a normal model.

(m) The following terminology will be useful in the sequel:

2.9 Definition. *Given an object A defined over $\mathcal{X}$ (a vector bundle, a subvariety, a cycle...), we will denote by A_η its restriction to X_F via η; the corresponding object A_η over the generic fibre, is called the restriction to the generic fibre of A.*

2.11 *Models and rational points.* We would like to understand the relations between models and rational points. We show now that, if we are in presence of a projective variety X_F over the function field F, the F rational points of X_F may be describe even more geometrically in terms of models and morphisms of k–varieties. Let $f : X_F \to \mathrm{Spec}(F)$ be a smooth projective variety over F.

We want to study an easy case first.

Suppose that X_F is a trivial variety; thus there is a k curve X_0 and an isomorphism $X_F \simeq X_0 \times_k \mathrm{Spec}(F)$. A natural model for X_F is then $p : X_0 \times B \to B$. Each time we have a k–morphism of curves $P : B \to X_0$, we can look to its graph $\Gamma_P : B \to X_0 \times_k B$.

The restriction of Γ_P to the generic point $\mathrm{Spec}(F)$ of B give rise to a point $P_F \in X_F(F)$.

Thus we get an inclusion $Hom_k(B; X_0) \subseteq X_F(F)$. Since X_0 is projective, this inclusion is indeed an equality:

2.11 Proposition. *If X_0 is projective then $Hom_k(B; X_0) = X_F(F)$.*

Proof: We need to prove that any point $p \in X_F(F)$ comes from a morphism $P : B \to X_0$. We first do the case when $X_0 = \mathbb{P}^N$. Fix coordinates on $\mathbb{P}^N_k$. A rational point $p \in \mathbb{P}^N(F)$ corresponds to $N+1$ rational functions $[f_0; \ldots; f_N]$ up to a non trivial scalar factor. By definition $[f_0; \ldots; f_N]$ defines a morphism from an affine open set of B to $\mathbb{P}^N$. Since B is smooth, this extends to a morphism from B to $\mathbb{P}^N$.

2.12 Exercise. In the proof of the proposition above we used the following fact: Let B be a smooth projective curve and U be a Zariski open set of it. Let $f : U \to \mathbb{P}^N$ be a morphism. Then f extends to a morphism $f' : B \to \mathbb{P}^N$. Prove it.

The general case is a consequence of the case of $\mathbb{P}^N$ and the Lemma below.

2.13 Lemma. *Let $f : \mathcal{X} \to B$ be a model of X_F. Suppose that $\mathcal{X}$ is a smooth k– variety. Then the morphism $f : \mathcal{X} \to B$ can be factorized as $f = p \circ i$ with $i : \mathcal{X} \to \mathbb{P}^N \times B$ a closed immersion and $p : \mathbb{P}^N \times B \to B$ is the second projection.*

Proof: Since $\mathcal{X}$ is projective over k; we can embed it inside some projective space $\iota \mathcal{X} \to \mathbb{P}^n$. The map $i := (\iota; f) : \mathcal{X} \to \mathbb{P}^N \times B$ has the searched properties.

The lemma above is useful because it will reduce the verification of many properties to the "easy" case of $\mathbb{P}^N_F$ with the corresponding trivial model. Observe that $\mathcal{X}$ is a closed subvariety of $\mathbb{P}^N \times B$. This is a special case of property (e) of 2.8. One should notice the following difference: here we started with a model $\mathcal{X}$ and then we constructed the embedding; in (e) of 2.8 we made the converse.

It is important to notice that the same proof applies in the general case:

2.14 Theorem. *Let X_F be a projective variety over F and $f : \mathcal{X} \to B$ be a projective model of it. Then there is a bijection:*

$$\{\text{Points}\, p \in X_F(F)\} \longleftrightarrow \{k - \text{morphisms } P : B \to \mathcal{X} \;\text{ s.t. }\; f \circ P = Id\}\,.$$

The theorem above give a geometric interpretation of F rational points of a projective variety: they correspond to k morphisms of the projective curve B to a model of the variety which composed with the structural morphism are the identity.

2.15 Remark. Observe that a rational point is the generic fibre of a section.

The following example shows that the hypothesis of projectiveness is essential.

2.16 Example. Let $F := k(t)$. The corresponding curve is $\mathbb{P}^1_k$. Consider the variety $X_F := \mathbb{A}^1_F$. Consider the point $p \in X_F(F)$ with coordinate t. This point *do not extend to a morphism* $P : \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{A}^1_k$. Indeed such a point will give a non trivial morphism from $\mathbb{P}^1_k$ to $\mathbb{A}^1_k$ and this is impossible.

2.18 *Integral points.* One of the main subjects of diophantine geometry, is the study of diophantine equations:

2.18 Example. Suppose that $R = k[t]$ and consider the equation

$$f(X) := X^n + a_{n-1}X^{n-1} + \dots a_0 = 0 \tag{2.18.1}$$

with $a_i \in R$. We may ask if there are solutions in R to such an equation. Observe that, since R is a normal ring, every solution of 2.18.1 which is in $k(t)$ is actually in R. Thus a solution of 2.18.1 is a factor of the polynomial $a_0(t)$. Consequently, up to the problem of factorizing $a_0(t)$, we can explicitly solve an equation of type "monic polynomial equal to zero". Observe that if $f(X)$ is not monic, the problem is already harder. The reader may try to find a procedure to find the solutions in R of an equation of the kind $f(X) = 0$ with f non monic.

If we consider systems of equations in more then one variable, the problem is more complicated and algebraic geometry helps. Before we start to deal with this problem we have to understand what a solution of an equation is.

In the example above, we were interested in solutions in $R = k[t]$ of a monic equation. Incidentally, due to the normality of R we could look for the solutions in the field $k(t)$ which coincide with the solutions in R. When the polynomial is not monic, there may be solutions in $k(t)$ which are not in R.

2.20 Example. Let

$$\begin{cases} F_1(t, X_1, \dots, X_n) = 0 \\ \vdots \\ F_m(t, X_1, \dots, X_n) = 0 \end{cases} \tag{2.20.1}$$

with $F_i(t, X_1, \dots, X_n) \in k[t][X_1, \dots, X_n]$, be a system of polynomial equations. An *integral solution* of 2.20.1 is a n–uple $(f_1(t), \dots, f_n(t))$ such that $F_i(t, f_1(t), \dots, f_n(t)) = 0$ for every i. Consider the ideal $I := (F_1(t, X_1, \dots, X_n), \dots, F_m(t, X_1, \dots, X_n)) \subset k[t][X_1, \dots, X_n]$. We can associate to an integral solution of 2.20.1 a morphism $P : k[t][X_1, \dots, X_n]/I \to k[t]$, thus a morphism of schemes $\mathbb{A}^1_k \to \mathrm{Spec}(k[t][X_1, \dots, X_n]/I)$. Conversely for every such a morphism we can construct an integral solution of 2.20.1. Observe that, via the natural inclusion $k[t] \hookrightarrow k[t][X_1, \dots, X_n]$ and the projection $k[t][X_1, \dots, X_n] \twoheadrightarrow k[t][X_1, \dots, X_n]/I$, the scheme $\mathrm{Spec}(k[t][X_1, \dots, X_n]/I)$ is naturally a scheme over $\mathrm{Spec}(k[t])$ and the morphism P is a section of the structural morphism.

We see that the example above is very similar to the interpretation of the rational points of a variety defined over a function field given in the previous section. We simply changed $\mathrm{Spec}(k(t))$ with $\mathrm{Spec}(k[t])$. It is natural then to expect an interpretation of the integral solutions in terms of sections and models.

With this in mind we can give the following

2.22 Definition. *Let U be a scheme and $f : \mathcal{X} \to U$ be a U–scheme. A U–integral point of $\mathcal{X}$ is a morphism $p : U \to \mathcal{X}$ such that $f \circ p = Id$. The set of the U–integral points of $\mathcal{X}$ is denoted by $\mathcal{X}(U)$.*

2.23 Remark. (a) If $U = \mathrm{Spec}(F)$ then a U integral point of X is a F–rational point of $\mathcal{X}$.

(b) The case we are interested in is when U is an open set of a smooth projective curve B. For instance in the example 2.20, U is an open set of $B = \mathbb{P}^1_k$. From now on, we suppose that U is a regular scheme of dimension at most one. In this case $S := B \setminus U$ is a divisor and we speak about S–integral points.

(c) Let $\eta : \mathrm{Spec}(F) \to U$ be the generic point of U. Then the restriction $\mathcal{X}_F$ of $\mathcal{X}$ to $\mathrm{Spec}(F)$ is a F–variety. The restriction to F of a U integral point of $\mathcal{X}$ give rise to a F–rational point. Thus we get an inclusion $\mathcal{X}(U) \subseteq \mathcal{X}_F(F)$.

(d) Suppose that $f : \mathcal{X} \to U$ is a projective morphism, then 2.14 tells us that the inclusion $\mathcal{X}(U) \subseteq \mathcal{X}_F(F)$ is in fact a bijection.

(e) The following example is important: Suppose that $U = \mathbb{A}^1_k \setminus \{a_1, \ldots, a_n\}$. An U–integral point of $\mathbb{A}^N$ is a N–uple $(f_1(t), \ldots, f_N(t))$, where $f_i(t) \in k(t)$ are such that the denominators of them are divisible only by $(t - a_j)$. Thus an U–integral point is a $F := k(t)$–rational point with "controlled" denominator.

Let F and B as before and $U \subseteq B$ be a Zariski open set. Denote $S := B \setminus U$.

Consider $X = \mathbb{A}^N_F$, then $X \subset \mathbb{P}^N$ and $\mathbb{P}^N \setminus \mathbb{A}^N$ is the divisor at infinity H_∞. Of course $\mathcal{X} := \mathbb{A}^N \times B$ and $\mathbb{P}^N \times B$ are models of $\mathbb{A}^N_F$ and $\mathbb{P}^N_F$. Observe that $\mathbb{A}^N \times B = \mathbb{P}^N \times B \setminus H_\infty \times B$ and $H_\infty \times B$, which is a divisor in $\mathbb{P}^N \times B$, is a model of $H_{\infty,F}$.

Let $P : U \to \mathcal{X}$ be an U integral point. It corresponds to an algebraic point $p \in \mathbb{P}^N(F)$ thus to a map $P : B \to \mathbb{P}^N \times B$. Since the image of U is contained in $\mathcal{X}$, we have that $P^*(H_\infty \times B$ is an effective divisor on B with support contained in S. Also the converse is true: if $p \in \mathbb{P}^N(F)$ is a point for which the corresponding section $P : B \to \mathbb{P}^N \times B$ is such that $P^*(H_\infty \times B)$ is a divisor with support contained in S, then p is an U integral point of $\mathcal{X}$.

2.24 Example. Suppose that $F = k(t)$, $B = \mathbb{P}^1$ and $U = \mathbb{A}^1$; thus $S = [1 : 0]$. A point P of $\mathbb{P}^1_F(F)$ is a couple of homogeneous polynomials $[f_1(x; y) : f_2(x; y)]$ having the same degree and no common factors. Suppose that P is an U integral point. By construction $P^*(H_\infty \times B)$ is the set of zeros of $f_2(x, y)$; thus P is an integral point if and only if $f_2(x; y) = y^n$; consequently P is nothing else then the choice of a polynomial as explained in the definition 2.22.

With this in mind the definition below become clear: Let $f : \mathcal{X} \to B$ be a flat projective morphism. Let $\mathcal{D} \hookrightarrow \mathcal{X}$ be an effective divisor.

2.25 Definition. *An $(\mathcal{D}; S)$–integral point of $\mathcal{X}$ is a point $p \in (\mathcal{X}_F)(F)$ such that, denoting by $P : B \to \mathcal{X}$ the corresponding section, we have that $P^*(\mathcal{D})$ is an effective divisor with support contained in S.*

Of course, if we take $\mathcal{Y} = \mathcal{X} \setminus \mathcal{D}$, a $(\mathcal{D}, S)$ integral point of $\mathcal{X}$ is a U–integral point of $\mathcal{Y}$.

We can give, without pain, variations of the definition above in the case of algebraic points. For instance

2.26 Definition. *Let L/F be a finite extension and B_L the corresponding curve. Let S_L be a closed set of B_L. A L point $p \in \mathcal{X}_F(L)$ is an $(\mathcal{D}, S_L)$–integral point if, denoting by $P : B_L \to \mathcal{X}$ the map corresponding to p, we have that $Supp(P^*(\mathcal{D})) \subset S$.*

As one can see the definition of $(\mathcal{D}, S)$ integral point is something which depends on the model of the variety defined over F and not only on the variety itself. Never the less we have the following

2.27 Proposition. *Suppose that $\mathcal{X}$ and $\mathcal{D}$ are as before. Let $h : \mathcal{X}' \to \mathcal{X}$ be a birational morphism and $\mathcal{D}' := h^*(\mathcal{D})$ then the set of $(\mathcal{D}; S)$–integral points and $(\mathcal{D}'; S)$–integral points coincide.*

The proof is left as exercise.

2.28 Remark. If we take $S = \emptyset$, then $(\mathcal{D}, \emptyset)$–integral points are points whose corresponding section *do not* intersects $\mathcal{D}$. If we take $\mathcal{D} = \emptyset$, then $(\emptyset, S)$ points are just F–rational points of $\mathcal{X}_F$. Thus the study of F–rational (or more generally L–rational) points is just a particular case of the study of $(\mathcal{D}, S)$ integral points.

3 The Kodaira Spencer class.

We will denote by X_L be the variety X_F, seen as a variety defined over L: by definition $X_L := X_F \times_{\mathrm{Spec}(F)} \mathrm{Spec}(L)$.

Within the varieties defined over F there are the varieties defined over k seen as varieties defined over F: if $X_k \to \mathrm{Spec}(k)$ is a variety, then $X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F) \to \mathrm{Spec}(F)$ is a variety defined over F. We will call such a variety a *k–trivial variety.*

Of course for this kind of varieties many questions have an easy answer or may be reduced to classical questions of algebraic geometry. For instance, since $X_k(k) \subseteq X_F(F)$ thus $X_F(F)$ is always infinite.

3.1 Exercise. Prove that $X_k(k)$ is a subset of $X_F(F)$.

There are varieties which are *not* k–trivial, but they become trivial after isomorphism or after a base change.

3.2 Example. Let $F = k(t)$. It is the field of fractions of $\mathbb{P}^1$. Consider the Curve $C := \{Y^2 = X^3 + t\}$. A priori it looks as a curve which is *not* defined over k but: consider the extension $L := F[t^{1/2}, t^{1/3}]$; over L the change of variables $X = t^{1/3}X_1$ and $Y = t^{1/2}Y_1$ give rise to an isomorphism of C with the curve $Y_1^2 = X_1^3 + 1$ which is defined over k.

3.3 Example. Again $F = k(t)$ and $X_F = \{(X + tY)^4 + Y^4 = 1\}$. Observe that $X_F(F)$ is an infinite set; indeed for every couple $(a, b) \in k^2$ such that $a^4 + b^4 = 1$ (there are infinitely many because k is algebraically closed), the couple $(a - tb, b)$ defines a point of X_F.

For this reason we give the following definition:

3.4 Definition. *Let X_F be a variety defined over F. We will say that it is* isotrivial *if there exists a variety X_k defined over k, a finite extension L of F and an L isomorphism*

$$X_L \simeq X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L).$$

Thus a variety is isotrivial if it is a trivial variety when we see it over the algebraic closure of F.

One would like to give some geometrical way to know if the variety is or not isotrivial.

Observe that, a variety defined over F is, roughly speaking a system of of polynomial equations where the coefficients of the polynomials are in F. Thus each coefficient may be seen as a function in some variables. If we give to each variable a value in k, we obtain a variety over k. For instance if $F = k(t)$ and $X_F = \{(X + tY)^4 + Y^4 = 1\}$, for each value of $t \in k$ we obtain a plane curve defined over k. Each time we specify the value of t, we obtain a quadric plane curve X_t; but the morphism $X = X_1 - tY_1$, $Y = Y_1$ induces an isomorphism of X_t with the curve $X_1^4 + Y_1^4 = 1$.

This can generalized to any isotrivial variety. Suppose that X_F is a k–trivial curve, then $X_k \simeq X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$. Thus every time we give specific values to the variable t, we obtain a k variety which is indeed isomorphic to X_k. Of course this argument may be generalized to any field F.

3.5 Exercise. Prove that, for every $a \in k^*$, the curves $Y^2 = X^3 + a$ are all isomorphic between them.

Thus we understand the following:

Isotrivial varieties are varieties defined over F which do not *move in moduli when we specify the parameters.*

We recall that, since F is a field of transcendence degree one over k, we may find an element $t \in F$ which is transcendental over k. Every other element of F is algebraic over $k(t)$.

3.6 *The Kähler differentials.* Let's recall the definition of the module of Kähler differentials $\Omega^1_{F/k}$ of F over K. It is a vector space over F of dimension one generated by the symbol dt. We have a differential map:

$$d : F \longrightarrow \Omega^1_{F/k}$$

defined in the following way: $d(t) = dt$; if $f \in F$ and $G(X) := \sum a_i(t)X^i$ is the minimal polynomial of f over $k(t)$ then define

$$d(f) =: \frac{-\sum a_i'(t) f^i}{\sum i a_i(t) f^{i-1}} dt.$$

Where if $a(t) \in k(t)$ then $a'(t) := \frac{d}{dt} \cdot a(t)$.

3.6 Exercise. Prove that:
– d is well posed (do not depend on the choices);
– d is k–linear;
– If $f, g \in F$ then $d(fg) = fd(g) + gd(f)$;
– If M is a F–vector space and $h : F \to M$ is a k–linear map such that $h(fg) = fh(g)+gh(f)$ then there exists a *unique* F–linear map $\tilde{h} : \Omega^1_{F/k} \to M$ such that $h = \tilde{h} \circ d$.

More generally, we recall from [Ha] the properties of the Kähler differentials of a scheme.

Let S be a scheme. Let $p : X \to S$ be a S–scheme. Let M be a quasi coherent sheaf over X. A S–derivation is a $\mathcal{O}_S$–linear morphism of sheaves

$$\psi : \mathcal{O}_X \longrightarrow M$$

such that $\psi(ab) = a\psi(b) + b\psi(a)$ (Liebnitz rule).

3.7 Definition. *There exists a quasicoherent sheaf $\Omega^1_{X/S}$, called the sheaf of Kähler differentials of X over S, with the following properties:*
(a) There is a S–derivation

$$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/S};$$

(b) For every S–derivation $\psi : \mathcal{O}_X \to M$ there exist a O_X–linear map $d_\psi : \Omega^1_{X/S} \to M$ such that $\psi = d_\psi \circ d$.

It is easy to prove that such a sheaf of Kähler differentials is unique up to isomorphism.

We will recall now the main properties of the Kähler differentials, the proofs may be found in [Ha] and [Ma] (or the reader may try to prove them, at least in some particular cases by exercise).

3.8 Theorem. *Let X, Y and S be schemes.*
*(a) (*First fundamental exact sequence*) Suppose we have a sequence of morphisms $Y \xrightarrow{f} X \to S$, then there is an exact sequence*

$$f^*(\Omega^1_{X/S}) \longrightarrow \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} \to 0.$$

*(b)(*Second fundamental exact sequence*) Suppose that, in the hypotheses of (a), the morphism $f : Y \to X$ is* a closed immersion*, Let $\mathcal{I}_Y$ be the ideal sheaf of Y in X then there is an exact sequence on Y*

$$\mathcal{I}_Y/_{\mathcal{I}_Y^2} \xrightarrow{d_Y} f^*(\Omega^1_{X/S}) \longrightarrow \Omega^1_{Y/S} \to 0;$$

where d_Y is defined as follows: let $\tilde{f} \in \mathcal{I}_Y/_{\mathcal{I}_Y^2}$, lift it to an element f in $\mathcal{I}_Y$, then $d_Y(\tilde{f})$ is the image in $f^*(\Omega^1_{X/S})$ of $d(f)$.

Observe that in (b), $\mathcal{I}_Y/_{\mathcal{I}_Y^2}$ has naturally a structure of $\mathcal{O}_Y$ module.

3.10 *Examples and properties of relative differentials.*

(a) $\Omega^1_{X/X} = 0$.

(b) It is easy to see that, if $U \overset{i}{\hookrightarrow} X$ is a open immersion then, $i^*(\Omega^1_{X/S}) = \Omega^1_{U/S}$.

(c) Let $X = \mathbb{A}^1_k := \mathrm{Spec}(k[t])$ then $\Omega^1_{X/k} = \mathcal{O}_X \cdot dt$ and, for every $f(t) \in k[t]$ we have $d(f) = f'(t)dt$.

(d) More generally, if $X = \mathbb{A}^n_k := \mathrm{Spec}(k[t_1, \dots, t_n]$ then $\Omega^1_{X/k} = \mathcal{O}_X \cdot dt_1 \oplus \dots \mathcal{O}_X \cdot dt_n$; for every $f \in k[t_1, \dots, t_n]$ we have that $d(f) = \sum_i \frac{\partial f}{\partial t_i} dt_i$.

(e) Yet more generally, Let X_1 and X_2 be two S–schemes; let $Z := X_1 \times_S X_2$, then $\Omega^1_{Z/S} = p_1^*(\Omega^1_{X_1/S}) \oplus p_2^*(\Omega^1_{X_2/S})$, $p_i : Z \to X_i$ being the projections.

(f) Suppose that $X := \{F = 0\} \subset \mathbb{A}^2_k$, is an affine plane curve, then by the second fundamental sequence,

$$\Omega^1_{X/k} = \mathcal{O}_X dt_1 \oplus \mathcal{O}_X dt_2 / (F_{t_1} dt_1 + F_{t_2} dt_2)$$

where $F_{t_i} := \frac{\partial F}{\partial t_i}$. In particular observe that, if $p \in X$ is a point where at least one of the partial derivatives of F do not vanishes (a smooth point) then the restriction of $\Omega^1_{X/k}$ to a neighborhood of p is free.

(g) One can show that X is a smooth k scheme if and only if $\Omega^1_{X/k}$ is locally free. If X is a k variety (it is locally of finite type) then $\Omega^1_{X/k}$ is of rank $\dim(X)$.

(h) The sheaf $\mathcal{I}_Y/_{\mathcal{I}_Y^2}$ is called the *conormal sheaf* of Y in X. Suppose that $X = \mathbb{A}^2$ and $Y = \{t_1 = 0\}$, then one easily see that $\mathcal{I}_Y/_{\mathcal{I}_Y^2}$ is the free sheaf generated by dt_1, while $\Omega^1_{Y/k}$ is the free sheaf generated by dt_2. Thus one can see the conormal sheaf as the sheaf of the infinitesimal displacements which are orthogonal to Y. Consequently, a global section of it will be an infinitesimal deformation of Y inside X.

(i) If $X = \mathrm{Spec}(F)$ where F is a field of transcendence degree n over k, then $\Omega^1_{X/k}$ is free of rank n and if $t_1, \dots, t_n$ is a transcendence basis of F over k, then a basis of $\Omega^1_{X/k}$ is $\{dt_1, \dots, dt_n\}$.

(j) Differentials commute to base change: Suppose we have a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S; \end{array}$$

where $Z = X \times_S S'$; then

$$\Omega^1_{Z/S'} = f^*(\Omega^1_{X/S}.$$

(k) Suppose that $f : X \to S$ is a smooth morphism. Then, by (g), $\Omega^1_{X/S}$ is locally free.

3.10 Definition. *The dual* $\mathcal{T}^1_{X/S} := Hom_{\mathcal{O}_X}(\Omega^1_{X/S}; \mathcal{O}_X)$ *is called the* relative tangent bundle *of* X *over* S. *A local section* $\delta \in \Gamma(U, \mathcal{T}^1_{X/S})$ *is, by definition a* S*–linear morphism* $\delta : \mathcal{O}_U \to O_U$ *such that* $\delta(ab) = a\delta(b) + b\delta(a)$ *and it is called* a derivation *of* X *over* S.

3.12 . Let $X_F \to \mathrm{Spec}(F)$ be a variety defined over F. Via the composition $X_F \to \mathrm{Spec}(F) \to \mathrm{Spec}(k)$, we may also see X_F as as scheme over k. Suppose from, from now on, that X_F is a smooth F–variety (this hypothesis is here just to simplify, many things may be done in a much general situation); by property (g) above, $\Omega^1_{X_F/F}$ is locally free of rank $\dim(X_F)$.

Let $\eta \to X_F$ be the generic point of X_F. The residue field of η is a field of transcendence degree $\dim(X_F)$ over F, thus it is of transcendence degree $\dim(X_F) + 1$ over k.

Since the composite of smooth morphisms is smooth, X_F is also a smooth k scheme (not of finite type in general), thus $\Omega^1_{X_F/k}$ is locally free. By property (j), it is locally free of rank $\dim(X) + 1$.

What we just remarked plus the first fundamental exact sequence allows to prove:

3.12 Theorem. *Let* F *be a field of transcendence* 1 *over* k. *Let* $f : X_K \to \mathrm{Spec}(F)$ *be a smooth* F *variety. Then, over* X_F *there is an exact sequence of locally free sheaves*

$$0 \to f^*(\Omega^1_{F/k}) \longrightarrow \Omega^1_{X_F/k} \longrightarrow \Omega^1_{X_F/F} \to 0 \tag{3.12.1}$$

Observe that:

(a) If t is a transcendence basis of F over k, then $f^*(\Omega^1_{F/k})$ is free generated by dt. The tangent bundle of $\mathrm{Spec}(F)$ over k is generated by an element δ such that $\delta(t) = 1$; let $a \in F$; and $F(X;Y) \in k[X;Y]$ is the mininal polynomial such that $F(a,t) = 0$ then $\delta(a) = -\frac{F_Y(a,t)}{F_X(a,t)}$.

3.14 Exercise. check that the definition make sense.

(b) The exact sequence is exact on the left because $f^*(\Omega^1_{F/k})$ is free of rank one and $\Omega^1_{X_F/k}$ is locally free.

(c) The tangent bundle of $\mathrm{Spec}(F)$ over k is a F–vector space of dimension one generated by $\frac{d}{dt}$.

The exact sequence defined in theorem 3.12 is one of the fundamental tools used in the study of the arithmetic of varieties defined over function fields (we hope that the reader will be convinced of this at the end of these lectures). In some way it must be seen as a canonical object attached to X_F *seen as a variety defined over a field of functions.*

Thus we give the following definition:

3.15 Definition. *Suppose that* $f : X_F \to \mathrm{Spec}(F)$ *is a smooth variety. Then the exact sequence 3.12.1 defines an element* $K(f)$ *in* $Ext^1(\Omega^1_{X_F/F}; f^*(\Omega^1_{F/k}))$. *This element is called* the Kodaira Spencer class of X_F over F.

3.16 Remark. Observe that the Kodaira Spencer class of $\mathbb{P}^N_F$ is zero.

3.17 Remark. (very important) If X_F is projective, then $Ext^1(\Omega^1_{X_F/F};\mathcal{O}_{X_F})$ is canonically isomorphic to the F finite dimensional vector space $H^1(X_F; f^*(\Omega^1_{F/k})\otimes(\Omega^1_{X/F})^\vee)$; thus the Kodaira Spencer class may be seen as:

(a) Either an element $K(f)\in H^1(X_F; f^*(\Omega^1_{F/k})\otimes(\Omega^1_{X/F})^\vee)$;

(b) or as a linear map of F vector spaces $K(f) : (\Omega^1_{F/k})^\vee \to H^1(X_F; (\Omega^1_{X/F})^\vee)$.

We will see that the Kodaira Spencer class of a variety over F *measure the non isotriviality of the variety.* It will be a cohomological invariant which vanishes if and only if the variety is isotrivial. Let's see some properties of the Kodaira Spencer class.

We start with a smooth projective variety $f : X_F \to \mathrm{Spec}(F)$. Denote by $K(f)\in H^1(X_F; f^*(\Omega^1_{F/k})\otimes(\Omega^1_{X/F})^\vee)$ the Kodaira Spencer class of X_F over F.

3.19 *k–trivial varieties.* Suppose that X_F is a k–trivial variety, then $K(f)=0$.

Indeed, there is a k–variety X_k and an isomorphism $X_F\simeq X_k\otimes_k \mathrm{Spec}(F)$. Thus, by property (e), $\Omega^1_{X_F/k} = p_1^*(\Omega^1_{X_k/k})\oplus p_2^*(\Omega^1_{F/k})$ and by property (j), $\Omega_{X_F/F} = p_1^*(\Omega^1_{X_k/k})$ thus the exact sequence 3.12.1 split and consequently $K(f)=0$.

3.20 *Examples of non isotrivial varieties.* To construct example of non isotrivial varieties one needs to use in general some knowledge of algebraic geometry. Usually the strategy is the following: one fix a smooth projective variety X_F over a function field F. We consider a model $f:\mathcal{X}\to B$ of it. If X_F is not isotrivial then, for general points p_1 and q_1 in B, the fibres X_{p_1} and X_{p_2} are *not* isomorphic.

3.20 Example. Consider the elliptic curve E over $F=k(t)$ given by the equation $Y^2 = X(X-1)(X-t)$. One can verify that the j invariant of E_p depends on p so E is not isotrivial.

3.21 Example. If C is a non hyperelliptic curve of genus three, then the map associated to the canonical linear system $H^0(C,K_C)$ is an embedding of C in $\mathbb{P}^2$. The image is a smooth curve of degree four. Any smooth curve of degree four in $\mathbb{P}^2$ is obtained in this way. If C_1 and C_2 are two such curves, then any isomorphism between them give rise to an isomorphism $H^0(C_1,K_{C_1})\to H^0(C_2,K_{C_2})$. This implies that two smooth curves of degree four in $\mathbb{P}^2$ are isomorphic if and only if there is a linear isomorphism of $\mathbb{P}^2$ sending one curve in the other. If C is a smooth curve of degree four, then the curves isomorphic to it are given by its orbit under the action of PGL_3 on $\mathbb{P}^2$. The curves of degree four of $\mathbb{P}^2$ are classified by the projective space of dimension 14 and the set of smooth curves is a open set inside it. and the group PGL_3 has dimension 9. Consequently given a curve of degree four, the space of curves isomorphic to it has dimension at most 9 thus we can find a curve of degree 4 which is not isomorphic to it.

Take a line in $\mathbb{P}^{14}$ which is not contained in the orbit of a fixed curve. The corresponding family of curves of genus three is not isotrivial.

A variation of this example, but which requires the knowledge of the theory of the Hilbert schemes shows that there exists non isotrivial curves of any genus.

Being isotrivial is not a birational invariant of a variety. The following example will show a non isotrivial surface birational to the projective plane (which is evidently isotrivial).

3.22 Example. Let $\ell_1, \dots, \ell_r$ be $r > 4$ general lines on $\mathbb{P}^2_k$. Each ℓ_i correspond to a point P_i of $\mathbb{P}^2(k(t))$. Let $\tilde{X} \to \mathbb{P}^2_{k(t)}$ be the surface obtained by blowing up these points. We claim that $\tilde{X}$ is not isotrivial.

It is easy to see that the fibre over a general point of B of a model of $\tilde{X}$ is the blow up of $\mathbb{P}^2$ on r points. Thus since the the lines are general, in order to prove that $\tilde{X}$ is isotrivial, it suffices to prove the following:

3.22 Proposition. *Let $\{p_1, \dots, p_r\}$ and $\{q_1, \dots q_r\}$ be two sets of points of $\mathbb{P}^2$; let X_1 be the surface obtained blowing up all the p_i's and X_2 be the surface obtained blowing up all the q_j's. Then X_1 is isomorphic to X_2 if and only if there is an element $\varphi \in PGL_3$ such that $\varphi(\{p_1, \dots, p_r\}) = \{q_1, \dots q_r\}$.*

Proof: Of course if there exists such a φ then X_1 is isomorphic to X_2. Let's prove the converse.

Suppose that there exists such an isomorphism $\varphi : X_1 \to X_2$. Denote by E_i the exceptional divisors of X_1, by F_j the exceptional divisors of X_2, by H_1 (resp. H_2) the pull back of the tautological bundle of $\mathbb{P}^2$ on X_1 (resp. X_2). If we show that, $\varphi^*(H_2) = H_1$ and for every j there exists a i such that $\varphi^*(F_j) = E_i$ we are done.

The canonical line bundle of X_1 (resp. X_2) is $K_1 = -3H_1 + \sum E_i$ (resp. $K_2 = -3H_2 + \sum F_j$).

Suppose that $\varphi^*(H_2) = dH_1 + \sum a_i E_i$ and $\varphi^*(F_j) = m_j H_1 + \sum_i n_{ji} E_i$. Since φ is an isomorphism, then $\varphi^*(K_2) = K_1$. Consequently, computations give: $\sum_j m_j = 3d - 3$ and $\sum_j n_{ji} = 1 - a_i$.

Since $(F_j; H_2) = 0$ we obtain $dm_j - \sum_i n_{ji} a_i = 0$. Summing up all the m_j we obtain

$$\begin{aligned} \sum_j m_j &= \frac{1}{d} \sum_j \sum_i a_i n_{ji} \\ &= \frac{1}{d} \sum_i a_i \sum_j n_{ji} \\ &= \frac{1}{d} \sum_i a_i (1 - a_i). \end{aligned}$$

Consequently $3d(d-1) = \sum_i a_i - a_i^2$ which gives $d = 1$ and $a_i = \pm 1$ or $a_i = 0$.

Since we have $(H_2; H_2) = 1$, we have $1 - \sum_i a_i^2 = 0$ thus $a_i = 0$. Consequently $\varphi^*(H_2) = H_1$. From the equality $(H_2; F_j) = 0$ we obtain then $m_j = 0$ and from the equality $(F_j; F_j) = -1$ we obtain $-\sum_i n_{ji}^2 = -1$ thus the conclusion follows.

3.25 *Base change.* Suppose that L/F is a finite field extension. Since we are in characteristic zero, $\Omega^1_{L/k} = \Omega^1_{F/k} \otimes_F L$.

Consider the L variety $f_L : X_L := X_F \times_F L \to \mathrm{Spec}(L)$. Its Kodaira Spencer Class $K(f_L)$ is an element in $H^1(X_L; f_L^*(\Omega^1_{L/k}) \otimes (\Omega^1_{X_L/L})^\vee)$. By cohomology of the base change, this vector space is $H^1(X_F; f^*(\Omega^1_{F/k}) \otimes (\Omega^1_{X/F})^\vee) \otimes L$; then via the natural inclusion $H^1(X_F; f^*(\Omega^1_{F/k}) \otimes (\Omega^1_{X/F})^\vee) \hookrightarrow H^1(X_L; f_L^*(\Omega^1_{L/k}) \otimes (\Omega^1_{X_L/L})^\vee)$, we obtain $K(f) = K(f_L)$.

3.25 Exercise. Prove this.

This means that the Kodaira Spencer class commutes to the base change of fields.

From 3.19 and 3.25 we get the important

3.26 Theorem. *Suppose that $f : X_F \to \mathrm{Spec}(F)$ is an isotrivial variety, then $K(f) = 0$.*

The important fact is that also the converse is true: we will prove that a variety is isotrivial if *and only if* its Kodaira Spencer class vanishes. This is very important because it give us a specific object in a specific vector space which can tell us if the variety is or not isotrivial.

3.27 Theorem. *Suppose that $g : Y \to \mathrm{Spec}(F)$ is a smooth projective* isotrivial *variety and $h : Y \to X_F$ is a dominant F–morphism; then the Kodaira Spencer class of X_F vanishes: $K(f) = 0$.*

3.28 Remark. The theorem above tells us essentially that if a variety over F is dominated by a variety defined over k, then itself is defined over k (up to extensions). Remark that we can develop a the theory of Kodaira Spencer class over fields of arbitrary characteristic; but in positive characteristic there are some caveat; for instance the theorem above is false

3.29 Exercise. (for the reader who knows positive characteristic) Let E_1 and E_2 be two elliptic curves defined over a field of characteristic $p > 0$. Suppose that α_p is a subgroup of E_i. Consider the family...

Proof: (Of 3.27) By 3.25 we may suppose that there is a projective k–variety Z and an isomorphism $Y \simeq Z \times_k F$. Over Y we have the following exact diagram:

$$\begin{array}{ccccccc}
0 \to g^*(\Omega^1_{F/k}) & \longrightarrow & h^*(\Omega^1_{X_F/k}) & \longrightarrow & h^*(\Omega^1_{X_F/F}) & \to 0 \\
\| & & \downarrow & & \downarrow & \\
0 \to g^*(\Omega^1_{F/k}) & \longrightarrow & \Omega^1_{Y/k} & \longrightarrow & \Omega^1_{Y/F} & \to 0 \\
 & & \downarrow & & \downarrow & \\
 & & \Omega^1_{Y/X_F} & = & \Omega^1_{Y/X} & \\
 & & \downarrow & & \downarrow & \\
 & & 0 & & 0. &
\end{array}$$

Where the first row is the pull back, via h, of the first exact sequence over X, the second row is the first exact sequence over Y (which, by hypotheses is a split exact sequence); the first column is the first exact sequence associated to the morphism h where we see X and Y as k–schemes; the second column is the first exact sequence associated to the morphism h where we see X and Y as F–schemes.

Observe that since X_F is smooth, $\Omega^1_{X_F/F}$ is a locally free vector bundle; in particular it is a flat $\mathcal{O}_{X_F}$–module. Thus the pull back to $Y$of an exact sequence of $\mathcal{O}_{X_F}$–modules with cokernel $\Omega^1_{X_F/F}$ is again an exact sequence (if $\Omega^1_{X_F/F}$ was not flat, since a priori h is not flat, the pull back of an exact sequence is just a complex). This implies that we have a natural map $h^* : Ext^1(\Omega^1_{X_F/F}; f^*(\Omega^1_{F/k})) \to Ext^1(h^*(\Omega^1_{X_F/F}); f^*(\Omega^1_{F/k}))$.

The diagram above shows that, the natural map

$$Ext^1(\Omega^1_{Y/F}; g^*(\Omega^1_{F/k})) \longrightarrow Ext^1(h^*(\Omega^1_{X/F}); g^*(\Omega^1_{F/k}))$$

send the Kodaira–Spencer class $K(h)$ to $h^*(K(f))$.

Thus the conclusion follows if we prove that the map

$$h^* : H^1(X_F; (\Omega^1_{X_F/F})^\vee \otimes f^*(\Omega^1_{F/k})) \longrightarrow H^1(Y; h^*((\Omega^1_{X_F/F})^\vee \otimes f^*(\Omega^1_{F/k})))$$

is injective.

3.30 Lemma. *Let $h : Y \to X$ be a dominant map between smooth varieties over a field of characteristic zero. Let E be a vector bundle over X. Then the natural map*

$$h^* : H^1(X; E) \longrightarrow H^1(Y; h^*(E))$$

is injective.

Proof: Since X and Y are both projective, by Bertini theorem, we may take a sufficiently generic hyperplane section of Y and thus we may suppose that h is a generically finite morphism.

Since h is finite, we have that $H^1(Y; h^*(E)) = H^1(X; h_*(\mathcal{O}_Y) \otimes E)$.

the natural map $h^* : H^1(X; E) \longrightarrow H^1(X; h_*(\mathcal{O}_Y) \otimes E)$ is obtained by tensorizing by E the exact sequence

$$0 \to \mathcal{O}_X \longrightarrow h_*(\mathcal{O}_Y) \longrightarrow B \to 0$$

and taking the H^1.

Since we are in characteristic zero, the trace map give a splitting of the exact sequence above. The conclusion follows.

3.32 . Before we continue with properties of the Kodaira Spencer class we need to understand the relations between the Kodaira Spencer Class and the models of a variety:

(a) Suppose that $f : \mathcal{X} \to B$ is a model of X_F, and suppose that $\mathcal{X}$ is a smooth k variety. Then we have an exact sequence

$$0 \to f^*(\Omega^1_{B/k}) \longrightarrow \Omega^1_{\mathcal{X}/k} \longrightarrow \Omega^1_{\mathcal{X}/B} \to 0. \tag{3.32.1}$$

By property (j) of differentials, the base change to $\mathrm{Spec}(F)$ of this exact sequence, is the exact sequence 3.12.1. Observe that, in general $\Omega^1_{\mathcal{X}/B}$ is not locally free.

(b) Suppose that X_F is smooth and projective and that $f : \mathcal{X} \to B$ is a model which is a smooth projective k–variety. The exact sequence 3.32.1 is an extension to which we can associate an element $K_{\mathcal{X}}(f) \in H^1(\mathcal{X}; (\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k}))$. The relation with the Kodaira Spencer class of $f : X_F \to \mathrm{Spec}(F)$ is the following:

The Leray spectral sequence associated to the morphism $f : \mathcal{X} \to B$ give rise to an exact sequence

$$\begin{aligned} 0 \to H^1(B; f_*((\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k}))) \to H^1(\mathcal{X}; (\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k})) \to \\ \to H^0(B; (\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k})) \to 0. \end{aligned}$$

Thus we may consider the image a of $K_{\mathcal{X}}(f)$ in $H^0(B; (\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k}))$. Taking the base change via the map $\eta : Spec(F) \to B$ we deduce a morphism

$$\alpha : H^0(B; (\Omega_{\mathcal{X}/B})^\vee \otimes f^*(\Omega^1_{B/k})) \otimes_k F \longrightarrow H^1(X_F; (\Omega^1_{X_F/F})^\vee \otimes f^*(\Omega^1_{F/k}))$$

Thus the image of a via this morphism is the Kodaira Spencer class $K(f)$. Observe that, since $\mathcal{X}$ and $\mathrm{Spec}(F)$ are both flat over B, the map α is injective; thus the Kodaira Spencer class of X_F is zero if and only if it is zero $K_{\mathcal{X}}(f)$. A similar property holds in a situation as in (h) of the properties of models 2.8.

3.34 . We will prove now that if a curve defined over F has vanishing Kodaira Spencer class, then it is isotrivial. The theorem holds for arbitrary smooth projective variety but the proof is more involved. This part is freely inspired by the notes [Gi].

The theorem we want to prove is the following:

3.34 Theorem. *Let $f : X_F \to \mathrm{Spec}(F)$ be a smooth projective curve over F. Then $K(f) = 0$ if and only if X_F is isotrivial.*

Observe that:

(1) By remark 3.16 we may suppose that the genus of X_F is at least one. We will give details when the genus is a least two and explain what one have to do in the genus one case.

(2) By 3.26 we need to prove that the vanishing of $K(f)$ implies the isotriviality of X_F.

The idea of the proof is the following: Suppose that the Kodaira Spencer class vanishes. If X_F is isotrivial then every time we specify the parameters, we will find always the same curve. This means that, we may fix a model of X_F and a point of B the fibre over which is smooth. That fibre will be the candidate for the isotrivial curve. Call such a curve X_0, we need to prove that, up to extension, X_F is isomorphic to $(X_0)_F$. We will show that the vanishing of the Kodaira Spencer class allows to construct such an isomorphism by successive approximations.

Before we start the proof, we would like to explain it in terms of differential algebra.

The field F is equipped with the relative tangent bundle $\mathcal{T}^1_{F/k}$, which is a vector space of dimension one. If we fix a transcendence basis t of F over k, we can find a generator $d \in \mathcal{T}^1_{F/k}$. The function $d : F \to F$ is a k–derivation. An important observation is the following:

3.35 Proposition. *Let L be the kernel of $d : F \to F$, then $L = k$.*

Proof: It is clear that $k \subseteq L$. If $F = k(t)$ then the proposition is clear. Suppose that F is a finite extension of $k(t)$. Let $y \in L$, there are polynomials $a_i(t) \in k[t]$ such that $\sum_{i=0}^n a_i(t)y^i = 0$. We may suppose that the n–uple of positive integral numbers $(\deg(a_0(t)), \ldots, \deg(a_n(t)))$ is minimal between all the polynomials with that property. Since $d(y) = 0$, we have that $\sum_{i=0}^n d(a_i(t))y^i = 0$ but since $\deg(d(a_i(t)) = deg(a_i(t))$ if and only if $deg(a_i(t)) = 0$ the conclusion follows.

The proposition above tells us that we can recover k from F and the derivation on it.

Suppose that $f : X_F \to \mathrm{Spec}(F)$ is a curve of genus bigger or equal then two with vanishing Kodaira–Spencer class. By definition the exact sequence

$$0 \to \mathcal{T}^1_{X_F/F} \longrightarrow \mathcal{T}^1_{X_F/k} \longrightarrow f^*(\mathcal{T}^1_{F/k}) \to 0$$

is split. Thus the derivation d over F lifts to a global derivation over X_F.

When we have a k algebra A equipped with a k derivation δ, we may consider the subring $A_0 := \{a \in A \ / \ \delta(a) = 0\}$ and we have a dominant morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$. This globalizes to k–schemes equipped with a global k derivation. Thus, from X_F and the global k–derivation on it, we may construct a k–curve X_0 with a k morphism $X_F \to X_0$. We will prove that, up to taking a finite extension of F, the morphism induces an isomorphism $X_F \simeq X_0 \times_k \mathrm{Spec}(F)$.

Observe that the *non vanishing* of the Kodaira–Spencer class, tells us exactly that the derivation d *do not* lift to a global derivation on X_F.

Proof: (of 3.34) Let t be an element of F which is transcendental over k. Without loss of generality we may suppose that there is a closed point p_0 which is a simple zero of t. The tangent bundle $\mathcal{T}^1_{F/k}$ is generated by a derivation $d : F \to F$ such that $d(t) = 1$. Let $\mathcal{O}_{B,p_0}$ be the local ring of p_0 over B and let $\widehat{\mathcal{O}}_{B,p_0}$ be its completion. Since t has a simple zero in p_0, the derivation d induces a k derivation $d : \mathcal{O}_{B,p_0} \to \mathcal{O}_{B,p_0}$. The derivation extends to a k–derivation $d : \widehat{\mathcal{O}}_{B,p_0} \to \widehat{\mathcal{O}}_{B,p_0}$.

Fix a model $f : \mathcal{X} \to B$ of the curve; up to changing t and p_0 if necessary, we may suppose that the fibre $X_0 := \mathcal{X}_{p_0}$ is smooth. Consider the $\widehat{\mathcal{O}}_{B,p_0}$–scheme $\mathcal{X}_0 := \mathcal{X} \times_B \mathrm{Spec}(\widehat{\mathcal{O}}_{B,p_0})$. We show now that it suffices to prove that $\mathcal{X}_0 \simeq X_0 \times_k \mathrm{Spec}(\widehat{\mathcal{O}}_{B,p_0})$. Indeed, since X_F has genus bigger then one, over the algebraic closure of F, the group of its automorphism is finite. We may take a finite extension of F and suppose that all the automorphisms of X_F are defined over F. Consider the set $Isom_{\overline{F}}(X_F; X_0 \times_k \mathrm{Spec}(F))$ of the isomorphisms between X_F and X_0 defined over the algebraic closure of F. This set, either it is empty or it has the same cardinality of $Aut(X_F)$. If K is an extension of the algebraic closure of F, denote by $Isom_K(X_F \times K; X_0 \times K)$ the set of isomorphisms between $X_F \times K$ and $X_0 \times K$ defined over K. We have a natural inclusion $Isom_{\overline{F}}(X_F; X_0 \times_k \mathrm{Spec}(F)) \hookrightarrow Isom_K(X_F \times \mathrm{Spec}(K); X_0 \times_k \mathrm{Spec}(K))$. Since they are set of the same cardinality, the two sets coincide. Consequently, suppose that there is an isomorphism of X_F and X_0 defined over the field of fraction of $\widehat{\mathcal{O}}_{B,p_0}$, then we can find an isomorphism between them defined over a finite extension of F.

3.36 Remark. The reduction above is where we used the fact that the the genus of the curve is bigger or equal then two. The same argument apply to a variety with finite set of automorphisms. The general case is more involved (it requires the use of the so called Artin Approximation theorem).

To prove the theorem we are reduced to prove the following:

3.37 Proposition. *Let $f : \mathcal{X} \to \mathrm{Spec}(k[\![t]\!])$ be a smooth projective curve defined over $\mathrm{Spec}(k[\![t]\!])$. Denote by X_0 the special fibre of $\mathcal{X}$ (the fibre over the closed point). Suppose that the Kodaira Spencer Class $K(f)$ vanishes. Then there is an isomorphism*

$$\mathcal{X} \simeq X_0 \times_k \mathrm{Spec}(k[\![t]\!]).$$

Remark that, *mutatis mutandis*, we can naturally define a Kodaira Spencer class in this contest.

Proof: Denote by k_n the artinian ring $k[\![t]\!]/t^{n+1}$ and by $\mathcal{X}_n$ the scheme $\mathcal{X} \times_{\mathrm{Spec}(k[\![t]\!])} k_n$. By induction on n we may suppose that there exists an isomorphism

$$i_n : \mathcal{X}_n \xrightarrow{\simeq} X_0 \times_k k_n;$$

and we will show that the vanishing of the Kodaira Spencer Class implies that there is an isomorphism $i_{n+1} : \mathcal{X}_{n+1} \simeq X_0 \times_k k_{n+1}$ whose reduction $mod(t^{n+1})$ is i_n.

In order to prove this we need to introduce the formal flow of a vector field. Let's describe it in the affine context: Let A be a ring with a derivation δ (we work in characteristic zero). Then the map

$$\begin{aligned}\exp(\delta) : & A \longrightarrow A[\![x]\!] \\ & a \longrightarrow \sum_{n=0}^{\infty} \frac{\delta^n(a)}{n!} x^n\end{aligned}$$

is a ring homomorphism. This is due to the fact that, iterating the Leibnitz rule, we obtain that

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a)\delta^{n-i}(b).$$

Since the definition of $\exp(\delta)$ is intrinsic, it globalizes to schemes: suppose that we have a scheme X with a global derivation δ, then there is a formal flow

$$\exp(\delta) : X \times \widehat{\mathbb{A}}_0 \longrightarrow X$$

characterized by the fact that

$$d(\exp(\delta))(\frac{d}{dx}) = \delta.$$

3.38 Exercise. Consider the maps $h_1 : X \times \widehat{\mathbb{A}}_0 \times \widehat{\mathbb{A}}_0 \to X \times \widehat{\mathbb{A}}_0$ and $h_2 : X \times \widehat{\mathbb{A}}_0 \times \widehat{\mathbb{A}}_0 \to X \times \widehat{\mathbb{A}}_0$ defined respectively as $h_1(a, x_1, x_2) := (a, x_1 + x_2)$ and $h_2(a, x_1, x_2) := (\exp(a, x_1), x_2)$. Prove that $\exp(\delta)(h_1) = \exp(\delta)(h_2)$.

The exercise above shows that the map $\exp(\delta)$ is the analogue of the flow of a vector field defined in differential geometry.

3.39 Proposition. *Let $\mathcal{X} \to \mathrm{Spec}k[\![t]\!]$ be a $k[\![t]\!]$–scheme. Suppose that $\mathcal{X}$ is equipped with a global derivation δ extending the natural derivation d on $k[\![t]\!]$. Denote by $\widehat{(\mathcal{X})}$ the formal completion of $\mathcal{X}$ with respect to the ideal (t) and by X_0 the k–scheme $\mathcal{X} \otimes_{k[\![t]\!]} k$. Then there is a canonical isomorphism*

$$\widehat{\mathcal{X}} \simeq X_0 \times_k \mathrm{Spec}(k[\![t]\!]).$$

Proof: Suppose that A is a $k[\![t]\!]$ algebra equipped with a derivation δ extending the natural derivation d of $k[\![t]\!]$, denote by $\widehat{A}$ the completion of A with respect to the ideal (t). Denote by A_0 the k–algebra $A/(t)$ and by A_n the algebra $A/(t^{n+1})$. Then, it suffices to prove that there is a sequence of canonical compatible isomorphisms

$$A_n \simeq A_0 \otimes_k k[\![t]\!]/(t^{n+1}).$$

Denote by ϕ the composite map

$$A \xrightarrow{\exp(\delta)} A[\![x]\!] \longrightarrow A_0[\![x]\!]$$

and by α the composite of ϕ with the reduction $A_0[\![x]\!] \to A_0$. Since $\exp(\delta)(t) = t+x$, we have that $\phi(t) = x$. Thus we obtain natural compatible maps $\phi_n : A_n \to A_0[\![x]\!]/(x^{n+1})$. It suffices to prove that each ϕ_n is an isomorphism. Of course ϕ_0 is an isomorphism, thus we may assume, by induction that ϕ_{n-1} is an isomorphism. Denote by I the ideal (t) of A and by J the ideal (x) of $A_0[\![x]\!]$. The map ϕ_n induces a map of A_0–modules $\varphi_n : I^n/I^{n+1} \to J^n/J^{n+1}$. It suffices to prove that, for every n, the map φ_n is an isomorphism of A_0 modules. Since the A_0–modules I^n/I^{n+1} and J^n/J^{n+1} are generated by the class of t^n and x^n respectively and $\phi(t^n) = x^n$, the conclusion follows.

Thus we may conclude the proof of 3.37 as follows: Since the Kodaira Spencer class of $\mathcal{X}$ vanishes, the derivation d on $k[\![t]\!]$ extends to a global derivation on $\mathcal{X}$. Thus, by proposition 3.39, there is an isomorphism

$$\widehat{\mathcal{X}} \simeq X_0 \times_k \mathrm{Spec}(k[\![t]\!]).$$

But since $\mathcal{X}$ is projective, by the formal GAGA, the isomorphism above is algebraic. Consequently, the conclusion follows.

4 The theory of heights.

In this section we will describe the theory of heights over function fields. Indeed this theory is only an interpretation of the intersection theory but one need to develop it in a language appropriate for the arithmetic purposes. Our aim is to keep the language and the methods very geometric but, in the meanwhile, to keep the analogy with the number fields case in mind. The dictionary between height theory over number fields and intersection theory over function fields is where the analogy number field/function fields has the best success. One should notice that this is the starting point of an entire theory: the Arakelov theory.

Heights theory over function fields is intersection theory over models. We want to introduce now some measure of the complexity of an algebraic point of a variety defined over a function field.

We start with a smooth projective variety X_F defined over our function field F. Denote by $\overline{F}$ the algebraic closure of F.

Let $P \in X_F(\overline{F})$. First of all, we see that there exists a minimal finite extension L/F such that $P \in X_F(L)$.

4.1 Definition. *We will denote by $F(P)$ such extension and by $[F(P) : F]$ the degree of $F(P)$ over F.*

The degree of $F(P)$ is a first measure of the complexity of the algebraic point P.

The point P is by definition a morphism $\mathrm{Spec}(\overline{F}) \to \overline{X}$. The field of definition of P is the minimal extension L of F, such that there is a point $\mathrm{Spec}(L) \to X_F \times_F \mathrm{Spec}(L)$ whose base change to $\overline{F}$ is P. Notice that the projection give rise to a map $\mathrm{Spec}(L) \to X_F$.

The second measure of complexity is the height and it depends on the choice of a line bundle on X_F.

We can extend the variety X_F to the algebraic closure $\overline{F}$ of F: $\overline{X} := X_F \times_F \mathrm{Spec}(\overline{F})$.

We introduce now the notion of "functions up to bounded functions":

4.2 Definition. *Let $F(\overline{X}(\overline{F});\mathbb{Z})$ be the group of functions from $\overline{X}(\overline{F})$ to $\mathbb{Z}$ and $B(\overline{X}(\overline{F});\mathbb{Z})$ be the subgroup of bounded functions (a function is bounded if the image is contained in $[n;m]$ for some integer n and m). We denote by*

$$H(\overline{X}(\overline{F});\mathbb{Z}) := \frac{F(\overline{X}(\overline{F});\mathbb{Z})}{B(\overline{X}(\overline{F});\mathbb{Z})}$$

the group of functions modulo bounded function. If $h : \overline{X}(\overline{F}) \to \mathbb{Z}$ is a function in $F(\overline{X}(\overline{F});\mathbb{Z})$, we denote by $h(\cdot) + O(1)$ its image in $H(\overline{X}(\overline{F});\mathbb{Z})$.

The height will be a function, depending on a line bundle, modulo bounded function.

Let L_F be a line bundle over X_F. First of all we introduce the notion of a model for the line bundle. Fix a model $f : \mathcal{X} \to B$ of X_F over B. In the sequel, we will denote by $\eta : X_F \to \mathcal{X}$ the natural inclusion.

4.3 Definition. *A model of L_F is a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that, if $\eta : X_F \to \mathcal{X}$ is the natural inclusion, then $\eta^*(\mathcal{L}) \simeq L_F$.*

4.4 Remark. (a) Suppose that X_F is a curve. The degree of the restriction of the line bundle $\mathcal{L}$ to the fibre over a closed point of B do not depend on the point. This number coincide with the degree of the line bundle L_F.

(b) Observe that a model of L_F is a line bundle on $\mathcal{X}$ whose restriction to the generic fibre is isomorphic to L_F.

(c) If X_F is $\mathbb{P}^N_F$ then a natural model of $\mathcal{O}(1)$ over the model $\mathbb{P}^N_k \times B$ is $p_1^*(\mathcal{O}(1))$ where $p_1 : \mathbb{P}^N_k \times B \to \mathbb{P}^N_k$ is the the first projection. We will denote this model, again by $\mathcal{O}(1)$.

It is very important to observe that models of line bundles always exist over suitable models of X_F:

4.5 Theorem. *Let X_F be a smooth projective variety over F and L_F be a line bundle over it. Let $f : \mathcal{X} \to B$ be a model of X_F. Then there is a blow up $\tilde{\mathcal{X}}$ of $\mathcal{X}$ and a model $\mathcal{L}$ of L_F over $\tilde{\mathcal{X}}$.*

Proof: Every line bundle L_F on X_F may be written as $L_F = M \otimes N^{\otimes -1}$ where M is generated by global sections and N is very ample (thus generated by global sections). Consequently it suffices to prove the theorem when L_F is generated by global sections.

Since L_F is generated by global sections, it defines a morphism $g_{L_F} : X_F \to \mathbb{P}^N_F$ for a suitable N; moreover $g^*_{L_F}(\mathcal{O}(1)) = L_F$. This morphism extends to a rational map $g : \mathcal{X} \dashrightarrow \mathbb{P}^N \times B$. Consequently there is a commutative diagram

$$\begin{tikzcd} & \mathcal{X}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{X}_1 \arrow[rr, dashed] & & \mathbb{P}^N_k \times B \end{tikzcd}$$

where the continuous arrows are morphisms. The variety $\mathcal{X}_1$ is birational to $\mathcal{X}$ and is it model of X_F over B. The line bundle $g^*(\mathcal{O}(1))$ is a model of L_F.

In order to give the definition of the height function, we need to generalize the relation between rational points and sections to algebraic points.

Let $P \in \overline{X}(\overline{F})$. Let $F(P)$ its field of definition. We can associate to $F(P)$ a smooth projective curve B_P and the natural inclusion $F \subset F(P)$ give rise to a finite covering $B_P \to B$.

Observe that, strictly speaking, the curve B_P is a model of the F–variety P. This model is actually, the unique model of it; this is due to the fact that there is a unique smooth model of a field of transcendence one. In in general the fact that there is more then one model, corresponds to the fact that, given a field of transcendence bigger then one, there is more then one smooth variety with that field as function field.

The same proof of 2.14 allows to prove the following:

4.6 Theorem. *Let L/F be a finite extension. Let $h : B_L \to B$ be the corresponding smooth projective curve and the corresponding finite morphism. There is a bijection:*

$$\{\text{Points}\, p \in X_F(L)\} \longleftrightarrow \{k-\text{morphisms}\ P : B_L \to \mathcal{X}\ \text{ s.t. }\ f \circ P = h\}.$$

We fix a smooth projective variety X_F over F and a line bundle L_F over it. The *height function with respect to L_F* is defined as following:

Fix a model $\mathcal{X} \to B$ of X_F and a model $\mathcal{L}$ of L_F over it.

The point P give rise to a k–morphism $P : B_P \to \mathcal{X}$. Thus we may consider the number

$$h_L(P) := \frac{1}{[F(P) : F]} \cdot \deg(P^*(\mathcal{L})).$$

4.7 Definition. *The height function with respect to L_F is the class in $H(\overline{X}(\overline{F}); \mathbb{Z})$ of the function $P \mapsto h(P)$ and it is denoted by $h_L(P) + O(1)$.*

Of course one must show that this definition do not depend on the choices: in particular we have to show that:

(a) it is independent on the model $\mathcal{L}$ of L_F;

(b) it is independent on the the model $\mathcal{X}$ of X_F;

(c) It do not vary if we consider the point P as defined over a bigger field.

We start by proving (a). For this we need a lemma which tells us about the shape of the different models of a line bundle L_F over X_F.

4.8 Definition. *Let B be a curve and $f : \mathcal{X} \to B$ be a morphism from a normal variety to B. Let D be a reduced irreducible divisor on $\mathcal{X}$:*

(i) D is said to be horizontal, *if $f|_D : D \to B$ is dominant;*

(ii) D is said to be vertical *if $f|_D : D \to B$ is a point.*

Observe that every divisor $D := n_i D_i$ has a *unique* decomposition $D = H + V$ with H which is a sum of horizontal divisors and V is a sum of a vertical divisors. The divisor H will be called the *horizontal part of* D and V will be called the *vertical part of* D. It is very important to notice that $D_\eta = H_\eta$. Indeed, the image on B of the generic point of an irreducible vertical divisor is a closed point.

The relation between two different models of a line bundle is resumed in the following.

4.9 Proposition. *Let X_F a smooth projective variety over F and let L_F be a line bundle over it. Let $f : \mathcal{X} \to B$ be a model of X_F and $\mathcal{L}_1$ and $\mathcal{L}_2$ be two models of L_F over it. Then there is a vertical Cartier divisor V over $\mathcal{X}$ such that*

$$\mathcal{L}_2 \simeq \mathcal{L}_1 \otimes \mathcal{O}_{\mathcal{X}}(V).$$

Proof: Consider $M := \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}$. By construction M is a model of $\mathcal{O}_{X_F}$. It suffices to prove that the models of $\mathcal{O}_{X_F}$ are $\mathcal{O}_{\mathcal{X}}(V)$ with V a vertical divisor.

First of all observe that $\mathcal{O}_{\mathcal{X}}(V)$ is a model of $\mathcal{O}_{X_F}$: we may suppose that V is effective. Indeed, write $V = V_1 - V_2$ with V_i effective. It suffices to prove that $\mathcal{O}_{\mathcal{X}}(V_i)$ is a model of $\mathcal{O}_{X_F}$. The restriction to the generic fibre of the global section section V of $\mathcal{O}_{\mathcal{X}}(V)$ is nowhere vanishing (indeed, the projection of V on B do not contains the generic point) thus the restriction to the generic fibre of $\mathcal{O}_X(V)$ is isomorphic to $\mathcal{O}_{X_F}$.

Let M be a model of $\mathcal{O}_{X_F}$. Write $M = \mathcal{O}_{\mathcal{X}}(D)$ for some divisor D. By hypothesis, D_η is principal, thus there is a meromorphic function $f \in F(X_F) = k(\mathcal{X})$ such that $D_\eta = (f)_\eta$. Since the decomposition in vertical and horizontal part is unique, $D_h = (f)_h$ thus $D = (f) + V$ with V a vertical divisor. From this we deduce that $M \simeq \mathcal{O}_{\mathcal{X}}(V)$.

The proposition above implies that the height function do not depend on the choice of the model of the line bundle:

4.10 Theorem. *Let X_F be a smooth projective variety over B and L_F be a line bundle over it. Let $f : \mathcal{X} \to B$ be a projective model of X_F over B and $\mathcal{L}_1$ and $\mathcal{L}_2$ be two models of L_F over it. Then there is a constant C for which the following holds:*

Let L/F be a finite extension and $p : \mathrm{Spec}(L) \to X_F$ be a L rational point. Let $h : B_L \to B$ be the finite covering corresponding to L and $P : B_L \to \mathcal{X}$ the morphism *associated to p; then*

$$\left| \frac{1}{[L:F]} \cdot \deg(P^*(\mathcal{L}_1)) - \frac{1}{[L:F]} \cdot \deg(P^*(\mathcal{L}_2)) \right| \leq C.$$

In particular the height function $h_{L_F}(\cdot)$(as a function modulo bounded functions) do not depend on the choice of the model of the line bundle L_F.

Proof: By hypothesis, we may find two effective vertical divisors V_1 and V_2 such that $\mathcal{L}_1(V_1) = \mathcal{L}_2(V_2)$. Thus we may suppose that there is an effective vertical divisor V such that $\mathcal{L}_1 = \mathcal{L}_2(V)$. We may find an effective divisor D on B such that $f^*(D) \geq V$.

Observe that $P^*(f^*(D)) = h^*(D)$ consequently $\frac{1}{[L:F]} \cdot \deg(P^*(f^*(D))) = \deg(D)$; thus $\frac{1}{[L:F]} \cdot \deg(P^*(f^*(D)))$ is independent on P.

The image of B_L is not contained in a fibre, consequently, for every effective vertical divisor V, we have that $\deg(P^*(V)) \geq 0$. Since $f^*(D) \geq V$, we have that $\frac{1}{[L:F]} \cdot \deg(P^*(f^*(D))) \geq \frac{1}{[L:F]} \cdot \deg(P^*(V)) \geq 0$. Since that $\deg(P^*(\mathcal{L}_1)) = \deg(P^*(\mathcal{L}_2)) + \deg(P^*(V))$ the conclusion follows.

Now we come to the proof of (b).

4.11 Proposition. *Let X_F be a smooth projective variety over F and L_F be a line bundle over it. Suppose that $(f_1 : \mathcal{X}_1 \to B; \mathcal{L}_1)$ and $(f_2 : \mathcal{X} \to B, \mathcal{L}_2)$ are two models of X_F and L_F respectively. Then there is a constant C for which the following holds:*

Let L/F be a finite extension and $p : \mathrm{Spec}(L) \to X_F$ be a L rational point of X_F; let $P_i : B_L \to \mathcal{X}_i$ be the corresponding sections. Then

$$\left| \frac{1}{[L:F]} \cdot \deg(P_1^*(\mathcal{L}_1)) - \frac{1}{[L:F]} \cdot \deg(P_2^*(\mathcal{L}_2)) \right| \leq C.$$

In particular the height function associated to L_F is independent on the choice of the models.

Proof: The two models are projective varieties with the same field of rational functions. Thus they are birational varieties. Consequently we may find a variety $\mathcal{X}_3$ and a diagram of birational morphisms

$$\begin{array}{ccc} & \mathcal{X}_3 & \\ \swarrow & & \searrow \\ \mathcal{X}_1 & & \mathcal{X}_2. \end{array}$$

Observe that $\mathcal{X}_3$ may be chosen to be a model of X_F. Thus we may suppose that there is a birational morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and $P_2 = g \circ P_2$. Consequently $\deg(P_2^*(\mathcal{L}_2)) = \deg(P_1^*(g^*(\mathcal{L}_2))$. Since $g^*(\mathcal{L}_2)$ are models of L_F over $\mathcal{X}_1$, the conclusion is a consequence of 4.10.

The proof of (c) is a consequence of this easy lemma (which we already implicitly used in the proof of 4.10):

4.12 Lemma. *Let L/F be a finite extension and $h : B_L \to B$ be the corresponding finite covering of curves. Let M be a line bundle on B then*

$$\frac{1}{[L:F]} \cdot \deg(h^*(M)) = \deg(M).$$

The proof is left to the reader.

The height of a rational point is the second measure of its complexity. As one can see from the definition, the computation of the height requires global information over the curve naturally associated to the field of definition of the point. This is why the height has an arithmetic nature.

4.13 Example. Let $X_F = \mathbb{P}^N_F$ and $L_F = \mathcal{O}(1)$. A natural model of it is $\mathcal{X} := \mathbb{P}^N_k \times B \to B$. A model of L_F is $p_1^*(\mathcal{O}(1))$, where $p_1 : \mathcal{X} \to \mathbb{P}^N_k$ is the natural projection. Let L/F be a finite extension and $p : \mathrm{Spec}(L) \to X_F$ be a rational point. Let $P : B_L \to \mathcal{X}$ the corresponding section. The composite of P with p_1 give rise to a map $T : B_F \to \mathbb{P}^N_k$. The height $h_{L_F}(p)$ of p is $\deg(T^*(\mathcal{O}(1))$.

Suppose that H is an hyperplane of $\mathbb{P}^N_k$; thus H is a global section of $\mathcal{O}(1)$. The fact that T do not factorize trough H is equivalent to say that the algebraic point p is not contained in H. In this case, the height of p is the degree of the effective divisor $T^*(H)$; thus it is lower bounded.

4.13 *Properties of heights.*

(a) The height function:

$$\begin{aligned} h_*(\cdot) : Pic(X_F) &\longrightarrow H(\overline{X}_F(\overline{F}); \mathbb{Z}) \\ L &\longrightarrow h_L(\cdot) \end{aligned}$$

is a morphism of groups (proof left to the reader).

(b) *Functoriality*: Let $g : X_F \to Y_F$ be a morphism of F–varieties. Let L_F be a line bundle on Y_F and $p\mathrm{Spec}(L) \to X_F$ be an algebraic point. Then

$$h_{g^*(L_F)}(p) = h_{L_F}(g(p)).$$

Observe that $g(p) := g \circ p : \mathrm{Spec}(L) \to Y_F$ is naturally a point in $Y_F(L)$.

Proof: Choose models $\mathcal{X}$ and $\mathcal{Y}$ of X_F and Y_F respectively. The morphism $g : \mathcal{X}_F \to Y_F$ extends to a rational map $g : \mathcal{X} \dashrightarrow \mathcal{Y}$ (why?). Thus we can find a variety $\mathcal{X}_1$ birational to $\mathcal{X}$ and a commutative diagram

$$\begin{tikzcd} & \mathcal{X}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{X}_1 \arrow[rr, dashed] & & \mathcal{Y} \end{tikzcd}$$

where the continuous arrows are morphisms, and not just rational maps. The variety $\mathcal{X}_1$ is a model of X_F, thus we may suppose that the morphism $g : X_F \to Y_F$ extends to a morphism $g : \mathcal{X} \to \mathcal{Y}$ between suitable models. The point p extends to a morphism $P : B_L \to \mathcal{X}$ and the point $g(p)$ extends to the morphism $g \circ P : B_L \to \mathcal{Y}$. We may suppose that there is a model $\mathcal{L}$ of L_F over $\mathcal{Y}$. Then $g^*(\mathcal{L})$ is a model of $g^*(L_F)$ over $\mathcal{X}$. The conclusion follows.

(c) Suppose that $D \in H^0(X_F; L_F)$ is a non zero global section. Then, for every point $p \in X_F(\overline{F})$ we have that

$$h_{L_F}(p) \geq O(1).$$

By the formula above we mean that, each time we fix a representative $h(\cdot)$ of the height function $h_{L_F}(\cdot)$, then there is a constant C such that $h(p) \geq C$ for every algebraic point p not in D.
Proof: Fix models $\mathcal{X}$ of X_F and $\mathcal{L}$ of L_F. The global section D extends to a global section $\mathcal{D} \in H^0(\mathcal{X}; \mathcal{L}(V)$ for some vertical divisor V. Observe that $\mathcal{L}(V)$ is a model of L_F. Let $p \in X_F \setminus D$. It extends to a section $P : B_p \to \mathcal{X}$. The fact that $p \notin D$ implies that the image of B_p is not contained in $\mathcal{D}$. We have that $P^*(\mathcal{D})$ is a non zero global section of $P^*(\mathcal{L}(V))$. Thus $\deg(P^*(L(V))$ is positive. The conclusion follows.

Property (c) has many formal consequences:

(d) Suppose that there is a power of L_F which is generated by global sections, then $h_{L_F}(\cdot) \geq O(1)$.
Proof: Since $h_{L_F^{\otimes n}}(\cdot) = nh_{L_F}(\cdot) + O(1)$, we may suppose that L_F is generated by global sections; this means that we can fix global sections $D_1, \ldots, D_m$ of L_F such that, for every $p \in X_F(\overline{F})$ there is D_i such that $p \notin D_i$. The conclusion follows from (c).

(e) If L_F is ample, then $h_{L_F}(\cdot) \geq O(1)$.

This is just a particular case of (d).

(f) Suppose that L_F is ample and M_F is another line bundle, then there is a constant A such that

$$h_{L_F}(\cdot) \geq Ah_{M_F}(\cdot) + O(1).$$

Proof: It suffices to remark that there is a positive integer n such that $L^{\otimes n} \otimes M^{\otimes -1}$ is ample.

(g) If L_F is an ample line bundle and M_F is a line bundle numerically equivalent to zero, then, for every $\epsilon > 0$ we have that

$$|h_{M_F}(\cdot)| \leq \epsilon h_{L_F}(\cdot) + O(1).$$

Proof: Since M_F is numerically equivalent to zero, by the Nakai Moishezon criterion, for every constant A, the line bundle $L_F \otimes M^{\otimes A}$ is ample. The conclusion follows from (f).

Properties (b)–(g) essentially tell us that, up to some error term, the height associated to a line bundle depends only on its numerical class. Beware that it is *not true* that the height depend only on the numerical class.

(h) Suppose that L is a line bundle and Y is the intersection of the base loci of $H^0(X_F, L_F^{\otimes n})$ for $n \in \mathbb{N}$, then for every $p \in (X_F \setminus Y)(\overline{F})$ we have $h_{L_F}(p) \geq O(1)$ where the involved constant is independent on p. The proof of this is again an application of (c) and left as exercise.

4.14 *Bounded families.* The most important property of height is that, if we can bound the height of a set of rational points on a projective variety, then this set is "controllable". In the best situation, this set will be finite.

4.14 Example. Let $F = k(t)$, thus $B = \mathbb{P}^1_k$, and $X_F = \mathbb{P}^1_F$. We take $L = \mathcal{O}(1)$. As model of X_F we take $p_1 : \mathcal{X} := \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ and as model of L we take $\mathcal{L} := p_2^*(\mathcal{O}(1))$, where $p_2 : \mathcal{X} \to \mathbb{P}^1_k$ is the second projection. Rational points $p \in X_(F)$ correspond to algebraic maps $\varphi_p : \mathbb{P}^1_k \to \mathbb{P}^1_k$ (project on the second factor).

Let $p \in X_(F)$ and $P : \mathbb{P}^1_k \to \mathcal{X}$ the corresponding section. Then

$$h_L(p) = \deg(\varphi_p(\mathcal{O}(1))) = \deg(\varphi_p).$$

Suppose that $S \subset X_F(F)$ is a set of rational points and suppose that it is a set of bounded height. This means that there exists a constant C such that, for every $p \in S$ we have that $h_L(P) \leq C$. Consequently, the corresponding set of maps φ_p, for $p \in S$ is a set of rational maps of bounded degree from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$.

Fix a positive integer n. The set of degree n maps from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ is in bijection with the set of lines in $H^0(\mathbb{P}^1_k; \mathcal{O}(n))$; thus it is in bijection with the k rational points of $\mathbb{P}^{n-1}_k$.

From the observation above, we deduce that there exists a variety Y over k and a natural injection $S \subset Y(k)$.

The example above can be generalized. We will see that the set of rational points of bounded height of a variety can be parametrized by the k–rational points of a variety defined over k. If the variety is not isotrivial then we can say even more: The set of F rational points of a non isotrivial variety is not Zariski dense (for simplicity here we will prove only the case of curves). This is very important, because it explain to us why we are interested on bounding the height of rational points.

We concluded the example above by saying that there is a "natural inclusion" $S \subset Y(k)$. We would now clarify what we mean by "natural inclusion". As usual this is done by looking to models.

4.15 Definition. *Let X and Y be two varieties over k. Suppose that $f_1 : X \to Y$ and $f_2 : X \to Y$ are two morphisms. We will say that f_1 is equivalent to f_2 are equivalent if there exist isomorphisms $\varphi : X \to X$ and $\psi : Y \to Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ \varphi\downarrow & & \downarrow\psi \\ X & \xrightarrow{f_2} & Y. \end{array}$$

Suppose that X and Y are projective varieties. Let U a quasi projective variety. A morphism $F : X \times U \to Y$ may be seen as a family of morphisms from X to Y parametrized by U. For every point $u \in U$ the map

$$\begin{aligned} F_u :& X \longrightarrow Y \\ & p \longmapsto F(p,u) \end{aligned}$$

is a morphism from X to Y.

4.16 Definition. *Let X and Y be projective varieties over k. Let S be a set of morphisms from X to Y. We will say that S is a* bounded family of morphisms *if there exists a quasi projective variety U a morphism $F : X \times U \to Y$ a subset $V \subseteq U(k)$ such that every $s \in S$ is isomorphic, as morphism, to a morphism F_u for some $u \in V$.*

Essentially, a bounded family of morphisms is a set of morphism which appear as a subset of a parametrized family of morphisms.

Of course the definition above has a counterpart for rational points of varieties:

4.17 Definition. *Let X_F be a variety defined over a function field F. Let $S \subseteq X_F(F)$ be a set of rational points. Fix a model $\mathcal{X} \to B$ of X_F. Let $\mathcal{S}$ be the set of sections $B \to \mathcal{X}$ deduced from S. We will say that S is* bounded *if the set $\mathcal{S}$ is a bounded family.*

A set of rational points is bounded if it can be parametrized by the set of points of a variety over k.

The first theorem about points of bounded heights is the following:

4.18 Theorem. *Let X_F be a projective variety and L be an ample line bundle over it. Suppose that S is a set of F rational points of X_F. Then S is bounded if and only if there is a constant C such that, for every point $p \in S$, we have that $h_L(p) \leq C$.*

Of course, when we say that the height is bounded, we mean that, for every representative of h_L, there is a constant depending on the representative which bound the representative of the height.

The proof of the theorem is similar to example 4.14, the main difficulty is that, for an arbitrary curve, there are many line bundles of fixed degree.

Before we start the proof we recall the definition of the Jacobian of a curve.

4.19 *The Jacobian of a curve.* Let B be a curve over an algebraically closed field k. We recall that there is an exact sequence

$$0 \to J(B)(k) \longrightarrow Pic(B) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Where $J(B)$ is a smooth projective group variety (an abelian variety) called the *Jacobian* of B. The dimension of $J(B)$ is equal to the genus $g(B)$. The set of closed points of $J(B)$ is in natural bijection with the set of isomorphism classes of line bundles of degree zero over B. For every integer d, the set of isomorphism classes of line bundles of degree d over B is in bijection with the set of rational points of a smooth projective variety $J^d(B)$ non canonically isomorphic to $J(B)$.

Let T be a k–scheme; and a line bundle L over $B \times T$ is said to be of relative degree d if for every field K and morphism $p : \mathrm{Spec}(K) \to T$ the line bundle $p^*(L)$ over the K curve $B \times_k K$ has degree d.

The variety $J^d(B)$ has the following universal property. There is a line bundle $\mathcal{P}^d \to B \times J^d(B)$ such that the following holds: suppose that we have a line bundle L of relative degree d over $B \times T$; then there is a morphism $f_L : T \to J^d(B)$ and a line bundle M on T such that $(id \times f_L)^*(\mathcal{P}^d) \simeq L \otimes p_2^*(M)$.

Proof: of 4.18 First of all observe that, by property (f) of heights, if the height of the elements of S, with respect of an ample line bundle L is bounded by a constant depending only on L and S (and the chosen representative of the height function), then for every line bundle M, the same property holds.

Suppose that S is bounded. Thus there is a model $\mathcal{X}$ of X_F, a quasi projective variety U and a morphism $F : B \times U \to \mathcal{X}$ such that, for every point $p \in S$ there is a point $u_p \in U(k)$ such that the section corresponding to p is isomorphic to $F(\cdot, u_p) : B \to \mathcal{X}$. We may suppose that U is irreducible.

We may compute the heights with respect to L by choosing an ample line bundle $\mathcal{L}$ on $\mathcal{X}$. The projection $p : B \times U \to U$ is smooth, in particular it is flat; consequently the degree of the restriction of $F^*(\mathcal{L})$ to the fibres of p do not depend on the fibre. The height of p with respect to L is the degree of $F(\cdot, u_p)^*(\mathcal{L})$, thus independent on p and in particular bounded.

We prove now that a set of F–rational points with bounded height is bounded.

Fix a model $p : \mathcal{X} \to B$ of X_F and a model $\mathcal{L}$ of L.

We may suppose that X_F is $\mathbb{P}^N$, $\mathcal{X} = B \times \mathbb{P}^N$ and $\mathcal{L} = \mathcal{O}(1)$. Indeed, since $\mathcal{L}$ is very ample, we may embed $\iota : \mathcal{X} \hookrightarrow \mathbb{P}^N$ in such a way that $\iota^*(\mathcal{O}(1)) = \mathcal{L}$. The claim follows.

Let $p \in S$. From the reductions above, either $h_L(p) = 0$ or $h_L(p) > 0$. The set of points p such that $h_L(p) = 0$ correspond to constant morphisms $P : B \to \mathbb{P}^N$; thus they are parametrized by a subset of $\mathbb{P}^N(k)$.

Thus we may restrict our attention to points p with $h_L(p) > 0$.

Taking a suitable Veronese embedding, we may suppose that for every point $p \in S$, we have that $h_L(p) \geq 2g(B) - 1$.

The theorem will be a consequence of the following proposition:

4.20 Proposition. *Let B a smooth projective curve of genus g. Let $d \geq 2g - 1$ and N be positive integers. Then the set of morphisms $f : B \to \mathbb{P}^N$ such that $\deg(f^*(\mathcal{O}(1))) = d$ is a bounded family.*

Proof: We recall that the set of morphism with the property stated is in bijection with the the set of couples $(V; L)$ where L is a line bundle of degree d and V is a subspace of

$H^0(B;L)$ of dimension $N+1$ and such that the natural restriction map $V \otimes \mathcal{O}_B \to L$ is surjective.

Let $J^d(B)$ and $\mathcal{P}^d$ be the variety and the line bundle which parametrize all the line bundles of degree d over B.

For every line bundle L of degree d on B, we have $h^0(B,L) = d+1-g$ (by hypothesis on the degree and Riemann–Roch), in particular the dimension of the space of global sections depends only on d. Consequently, if $p : B \times J^d(B) \to J^d(B)$ is the projection, we have that $p_*(\mathcal{P}^d)$ is a *vector bundle* of rank $d+1-g$.

Since every line bundle of degree d on B is generated by global sections, we have a surjective morphism

$$p^*(p_*(\mathcal{P}^d)) \longrightarrow \mathcal{P}^d \to 0.$$

Let $g : Gr^N(p_*(\mathcal{P}^d)) \to J^d(B)$ be the relative grassmanniann of sub bundles of rank N of $p_*(\mathcal{P}^d)$. Over $Gr^N(p_*(\mathcal{P}^d))$ we have a universal subbundle $\mathcal{N} \hookrightarrow g^*(p_*(\mathcal{P}^d))$. Since $Gr^N(p_*(\mathcal{P}^d)) \times_k B = Gr^N(p_*(\mathcal{P}^d)) \times_{J^d(B)} (J^d(B) \times_k B)$, if we denote by q_1 the projection $q_1 : Gr^N(p_*(\mathcal{P}^d)) \times_k B \to Gr^N(p_*(\mathcal{P}^d))$ and by q_2 the projection $q_2 : Gr^N(p_*(\mathcal{P}^d)) \times_k B \to B$, we have a natural morphism over $Gr^N(p_*(\mathcal{P}^d)) \times_k B$

$$\varphi : q_1(\mathcal{N}) \longrightarrow q_2^*(\mathcal{P}^d).$$

By the universal property of $J^d(B)$ and of the relative grassmannian, every couple (V,L) with L line bundle of degree d on B and V subspace of dimension N of $H^0(B;L)$ corresponds to a closed point of $Gr^N(p_*(\mathcal{P}^d)$. The couples which correspond to morphisms $B \to \mathbb{P}^N$ of degree d, are the couples for which the natural map $V \otimes \mathcal{O}_B \to L$ is surjective.

Let Q be the cokernel of φ and Z be the support of Q. The lemma of Nakayama implies that Z intersect the fibre of $q_1 : Gr^N(p_*(\mathcal{P}^d)) \times_k B \to Gr^N(p_*(\mathcal{P}^d))$ over the point $(V,L) \in Gr^N(p_*(\mathcal{P}^d))$ if and only if the morphism $V \otimes \mathcal{O}_B \to L$ is not surjective. Let U be the open set complementary to the image of Z via q_1. It is non empty because we know that there exist morphisms of degree d of B in $\mathbb{P}^n$. The morphisms of B of degree d in $\mathbb{P}^N$ are in bijection with the set of closed points of U. The conclusion follows.

When X_F is a non isotrivial family we can say even more. The existence of too many F–rational points of bounded height implies that the Kodaira Spencer class of X_F vanishes. Observe that if X_F is trivial then, surely all the points "coming from k" have bounded height. Thus the existence of many points of bounded height is essentially equivalent with the fact that the variety is isotrivial.

Proposition 4.20 tells us that there is a quasi projective variety which classifies morphisms of fixed degree from a projective curve to the projective space. Before we can study rational points of bounded height over an arbitrary non isotrivial variety we need to prove that there is a variety classifying morphisms of fixed degree from a smooth projective curve to an arbitrary variety. Usually one can deduce the existence of such a variety appealing to the theory of the Hilbert schemes. Here, for sake of self completeness we will deduce it from 4.20 and some considerations in projective geometry.

4.21 Theorem. *Let X be a smooth projective variety, L be an ample line bundle over it and d a positive integer. For every curve B, the set*

$$S := \{f : B \to X \ / \ \deg(f^*(L)) = d\}$$

is a bounded family.

Proof: First of all we may suppose that L is very ample, thus it induces an inclusion $i : X \hookrightarrow \mathbb{P}^N$. Consequently the set S is a subset of the set $S_{\mathbb{P}}$ of morphisms from B to $\mathbb{P}^N$ of degree d. Since, by proposition 4.20, $S_{\mathbb{P}}$ is the set of closed points of a quasi projective variety U, it suffices to prove that S is the set of closed points of a closed subvariety of U.

This will be a consequence of the following two lemmas:

4.22 Lemma. *Let Y be a smooth projective variety and $Z \hookrightarrow Y$ be a closed subscheme. Let $F : U \times B \to Y$ be a morphism and $q \in B(k)$ be a closed point. Then there exists a closed subscheme $N_q \hookrightarrow U$ with the following property: let $u \in U(k)$ then $F(u, q) \in Z$ if and only if $u \in N_q(k)$.*

4.23 Lemma. *Let Y be a smooth projective variety $Z \hookrightarrow Y$ be a closed subscheme, L an ample line bundle on Y and d a positive integer. Then there exists a positive integer a with the following property: let $f : B \to Y$ be a morphism such that:*

– $\deg(f^*(L)) = d$;

– *there exist a distinct points $p_1, \ldots, p_a \in B(k)$ such that $f(p_i) \in Z$.*

Then $f(B) \subseteq Z$.

We show how the two lemmas imply the theorem: appy Lemma 4.23 to $Y = \mathbb{P}^N$ and $Z = X$. Let a be the number deduced from the lemma. Fix a distinct points $q_1, \ldots, q_a \in B(k)$. Apply lemma 4.22 to the variety U constructed in proposition 4.20 and the points q_i. We deduce the existence of a subvarieties $N_{q_1}, \ldots, N_{q_a}$. Let $N := N_{q_1} \times_U \cdots \times_U N_{q_a} \hookrightarrow U$. The scheme N is the schematic intersection of the N_{q_i}'s. Consequently, $u \in N(k)$ if and only if $F(q_i, u) \in Z$ for every i. By lemma 4.23, the morphism $F(\cdot, u) : B \to \mathbb{P}^N$ factorizes through X if and only if $u \in N(k)$. The conclusion follows.

Proof: of 4.22: Denote by i_q the inclusion $U \to U \times B$ obtained by $u \mapsto (u, q)$ and by F_q the morphism $F \circ i_q : U \to Y$. Let N_q be the variety obtained by the cartesian diagram

$$\begin{array}{ccc} N_q := U \times_Y Z & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{F_q} & Y. \end{array}$$

A simple diagram chasing shows that N_q has the searched properties.

Proof: of 4.23: By induction on the codimension, we may suppose that Z is an effective divisor of Y. We may find a effective smooth ample divisor D and a positive integer n

such that $L^n = \mathcal{O}_Y(Z+B)$. Take $a = nd+1$. Let $f : B \to Y$ be a morphism then $\deg(f^*(\mathcal{O}(Z))) \leq nd$ because $\deg(f^*(\mathcal{O}(Z)) + \deg(f^*(\mathcal{O}(D)) = nd$ and D is ample. Suppose that $f(B) \not\subset Z$, then $\deg(f^*(Z)) \geq a$. The conclusion follows.

Unfortunately we cannot prove that a variety with a dense set of rational points with bounded height is isotrivial. We can prove it in a very interesting situation.

4.24 Definition. *We will say that a variety X over a field is* general *if it do not contains rational curves.*

4.25 Remark. A general variety *is not* a variety of general type in general. For instance an abelian variety is general from this point of view.

The main theorem of this subsection is the following which tells us that a non isotrivial projective general variety cannot have a Zariski dense set of rational points of bounded height.

4.26 Theorem. *Let X_F be a smooth projective general variety defined over F and L be an ample line bundle over it. Suppose that there is a constant A such that the set*

$$X_F(F)_{\leq A} := \{p \in X_F(F) \ \ / \ \ h_L(p) \leq A\}$$

is Zariski dense. Then the Kodaira Spencer class of X_F vanishes.

4.27 Remark. By 3.34 this implies that, if a curve has an infinite set of rational points with bounded height, then it is isotrivial.

Proof: We may fix a smooth and projective model $f : \mathcal{X} \to B$ of X_F and we may suppose that L has an ample model $\mathcal{L}$ over $\mathcal{X}$. The set S extends to a set of sections

$$\mathcal{S} := \{P : B \to \mathcal{X} \ \ / \ \ f \circ P = Id\}.$$

Taking a suitable subset of $\mathcal{S}$ we may suppose that:

(a) For every $P \in \mathcal{S}$, we have that $\deg(P^*(\mathcal{L})) = d$, for a suitable positive integer d.

(b) The union of the images of the P's for $P \in \mathcal{S}$ is dense in $\mathcal{X}$.

Point (b) is due to the fact that S is Zariski dense on the generic fibre.

By theorem 4.21 the set $\mathcal{S}$ is a bounded family, thus there is a quasiprojective variety U and a commutative diagram

$$\begin{array}{ccc} B \times U & \xrightarrow{G} & \mathcal{X} \\ & \searrow \quad \swarrow & \\ & B. & \end{array}$$

The first oblique arrow is the natural projection, second is f and G is dominant. Let $\overline{U}$ be a smooth compactification: the variety $\overline{U}_F := \overline{U} \times \mathrm{Spec}(F) \to \mathrm{Spec}(F)$ is an isotrivial variety. We can find a blow up $\tilde{U}$ of $\overline{U}_F$ with exceptional divisor E and a dominant map $\tilde{G} : \tilde{U} \to \mathcal{X}$. Since X_F is general and E is covered by rational curves, $\tilde{G}(E)$ is contained in a vertical divisor of $\mathcal{X}$. Since the Kodaira Spencer class of $\tilde{U}$ vanishes on $\tilde{U} \setminus E$, the conclusion follows by 3.27.

It interesting to generalize the statement before a little bit: Recall that a line bundle M on a smooth projective variety is said to be *big* if there is a positive integer n, an ample divisor L and an effective divisor D such that $M^{\otimes n} = L(D)$.

4.28 Proposition. *Let X_F be a smooth projective variety over F and M be a big divisor on it. Suppose that there is a constant A and the set*

$$\{p \in X_F(F) \ \ / \ \ h_M(p) \leq A\}$$

is Zariski dense. Then the Kodaira Spencer class of X_F vanishes.

The proof is left as exercise.

The following exercise shows that the hypothesis that X_F do not contain rational curves is essential.

4.29 Exercise. (a) Let $\tilde{X}$ be the surface defined in the example 3.22 prove that the set of points with bounded height is Zariski dense: an infinite family of such points is the family of lines in $\mathbb{P}^2_k$ different to the lines ℓ_i.

(b) (A non rational example) Take F to be the field of rational functions of a curve B having genus at least two without automorphisms. Let X_F be the surface $(B \times B)_F$. Let $\Delta : B \to B \times B$; It corresponds to a point $P_\Delta \in X_F(F)$. Let $\tilde{X}_F \to X_F$ be the blow up of X_F in P_Δ.

– Prove that the Kodaira Spencer class of X_F is different from zero.

For each point $p := (a, b) \in B \times B(k)$, consider the map $\varphi_p : B \to B \times B$ given by $x \mapsto (a, b)$.

– Prove that if $a \neq b$ then φ_p corresponds to a point $P_p \in \tilde{X}_F(F)$ with bounded height.

– Prove that the P_p are dense in $\tilde{X}_F$.

(c) Let E_1 and E_2 be two elliptic curves over k which are not isogenous and such that $End(E_i) = \mathbb{Z}$. Let B be a curve with two non trivial morphisms $\varphi_i : B \to E_i$. Let F be the function field of B. Consider the points $P_0 := (\varphi_1, \varphi_2)$, $P_1 = (\varphi_1, 0)$ and $P_2 := (0, \varphi_2)$ over the surface $(E_1 \times E_2)_F$ over F. Let $\tilde{Y} \to (E_1 \times E_2)_F$ be the blow up over the points P_i's.

– Prove that $\tilde{Y}$ is not isotrivial.

– For every point $q \in E_1 \times E_2(k)$ let $P_q := P_0 + q$. Prove that if q is torsion then the height (with respect to a symmetric line bundle) of p_q is the same then the height of P_0.

– Deduce that the set of points of bounded height on $\tilde{Y}$ is Zariski dense.

5 Logarithmic differentials.

When we consider a variety X_F with a divisor D over it and we want to study (D,S) integral points over it. we may think the divisor D as the divisor at infinity of the variety. An important tool in this case is the the sheaf of *differentials with logarithmic poles around* D. First of all we suppose that D is a simple normal crossing divisor:

5.1 Definition. *Let X be a variety and D an effective divisor over it. We will say that D is* simple normal crossing *(snc for short) if $D := \sum D_i$ with D_i smooth and, for every closed point $p \in X$, denoting by $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$ and by $\{x_1, \ldots, x_n\}$ a set of generators of $\mathfrak{m}_p/\mathfrak{m}_p^2$ as $\mathcal{O}_{X,p}/\mathfrak{m}_p$–vector space, there is an integer $r < n$ such that the restriction of D to* $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,p})$ *is $x_1 \cdot x_2 \cdots x_r = 0$.*

Simple normal crossing divisors are important because we can define the sheaf of differentials with logarithmic poles around them. We recall the main definition (cf. [Dl chap. II.3] for details and more)

Let X be a variety (over any field k) and D be a snc divisor over it. Denote by U the open set $X \setminus D$ and by $\iota : U \to X$ the inclusion. the sheaf $\iota_*(\Omega^i_U)$ coincide with the sheaf of differentials of order i over X with poles along D. The de Rham complex on U induces a map $d : \iota_*(\Omega^1_U) \to \iota_*(\Omega^2_U)$.

5.2 Definition. *The sheaf $\Omega^1_{X/k}(\log(D))$ of differentials with logarithmic poles around D is the subsheaf of $\iota_*(\Omega^1_U)$ locally generated by ω such that ω and $d(\omega)$ have at most a simple pole around D.*

We resume here some properties of the logarithmic differentials:

(a) The sheaf of logarithmic differentials is locally free of rank $n = \dim(X)$; more precisely, suppose that $D = \sum D_i$, for every $p \in X$ choose local parameters $f_1, \ldots, f_n$ in such a way that locally around p the divisor D_i is given by $f_i = 0$ $(i = 1, \ldots r)$; then a local basis for $\Omega^1_{X/k}(\log(D))$ is given by $\frac{df_1}{f_1}, \ldots, \frac{df_r}{f_r}, df_{r+1}, \ldots, df_n$.

(b) There is a natural inclusion $\Omega^1_{X/k} \hookrightarrow \Omega^1_{X/k}(\log(D))$. The quotient is canonically isomorphic to $\mathcal{O}_D$; the map $\Omega^1_{X/k}(\log(D)) \to \mathcal{O}_D$ is called the residue map.

(c) The line bundle $\bigwedge^n \Omega^1_{X/k}(\log(D))$ is $K_{X/k}(D)$ the line bundle of differential forms of maximal degree with at most simple poles around D.In particular, if X is a curve, then $\deg(\Omega^1_{X/k}(\log(D))) = 2g(X) - 2 + \deg(D)$. Observe that, in this case, $\deg(\Omega^1_{X/k}(\log(D)))$ is the Euler characteristic of the affine curve $U := X \setminus D$.

(d) Suppose that B is a curve, $F = k(B)$, and $f : X \to B$ is a flat projective morphism. Then the restriction D_η to the generic fibre X_η of D is a snc divisor. The natural inclusion (coming from property (b) above and theorem 3.8 (a)) $f^*(\Omega^1_{B/k}) \hookrightarrow \Omega^1_{X/k}(\log(D))$ restricted to the generic fibre give rise to an extension of vector bundles

$$0 \to f^*(\Omega^1_{F/k}) \longrightarrow (\Omega^1_{X/k}(\log(D)))_\eta \longrightarrow \Omega^1_{X_\eta/F}(\log(D_\eta)) \to 0 \qquad (5.3.1).$$

(e) Suppose that Y is a curve and $h : Y \to X$ is a morphism whose image is not contained in D, then the pull back to Y of a differential with logarithmic pole around D is a differential with a logarithmic pole around the support of $h^*(D)$. Thus we get a map

$$h^*(\Omega^1_{X/k}(\log(D))) \longrightarrow \Omega^1_{Y/k}(|h^*(D)|)$$

where $|h^*(D)|$ is the reduced divisor with the same support then $h^*(D)$.

(f) Suppose that X_F is a smooth projective variety defined over F and D_F is a simple normal crossing divisor over it. Fix a model $f : \mathcal{X} \to B$ of X_F and $\mathcal{D}$ of D_F. By Hironaka theorem of resolution of singularities, we may take a blow up of $\mathcal{X}$ and suppose that D_F has a model which is simple normal crossing. In particular, by 2.27, when we want to study $(\mathcal{D}; S)$–integral points, we may always suppose that $\mathcal{X}$ is smooth and $\mathcal{D}$ is simple normal crossing.

(g) Suppose that we are in the situation of (d). Then the exact sequence 5.3.1 allows to define a Kodaira Spencer class $KS_{(X_\eta, D_\eta)} \in Ext^1(\Omega^1_{X_\eta}(\log(D_\eta)); f^*(\Omega^1_{F/k}))$. Observe that, by functoriality, the inclusion in (b) give rise to a map

$$\alpha : Ext^1(\Omega^1_{X_\eta}(\log(D_\eta)); f^*(\Omega^1_{F/k})) \to Ext^1(\Omega^1_{X_\eta}; f^*(\Omega^1_{F/k}))$$

and $\alpha(KS_{(X,D)}) = KS(f)$. In particular, if the latter is not zero, the former cannot vanish. If the latter vanish, then it may happens that the former do not vanish; this corresponds to a family of divisors on a fixed variety parametrized by B.

6 First approaches to ABC .

At this point we can state the main conjecture which, in case of a positive solution will solve more or less all the qualitative diophantine problems over function fields. Using the analogy between number fields and function fields, an analogue conjecture can be stated over number field, but, in this case, the solution of it is even more far away.

Before we state the conjecture we need another definition.

6.1 Definition. *Let X be a smooth projective variety and D an effective divisor on it. Let Y be a smooth projective curve. If $f : Y \to X$ is a morphism such that $f(Y) \not\subseteq D$. Suppose that $f^*(D) = \sum n_i P_i$. Then we define*

$$N^{(1)}_D(f) = \sum_i \min\{1, n_i\}.$$

Observe that $\chi(Y) + N^{(1)}_D(f) = \chi(Y \setminus \{f^{-1}(D)\})$.

The main conjecture is the following:

6.2 Conjecture. *Let B be a smooth projective curve and $\mathcal{X}$ a smooth projective variety. Let D be a simple normal crossing divisor on $\mathcal{X}$. Suppose that $f : \mathcal{X} \to B$*

is a flat morphism. Then, for every $\epsilon > 0$ there is a constant C and a proper closed subvariety $Z \subsetneq \mathcal{X}$ such that the following holds: for every curve smooth projective curve Y with a finite morphism $h : Y \to B$ and for every morphism $P : Y \to \mathcal{X}$ whose image is not contained in $D \cup Z$ we have that

$$\deg(P^*(K_X(D))) \leq (1+\epsilon)(\chi(Y) + N_D^{(1)}(P)) + C\deg(h).$$

The conjecture is about the couple (X_η, D_η) defined over the function field F and can be stated as follows there. First of all one have to notice that, up to bounded functions, the function $\frac{N_D^{(1)}(P)}{[F(P):F]}$ do not depend on the chosen model.

6.3 Proposition. *Suppose that X_F is a variety over F and D is an effective divisor over it. Let $f_i : \mathcal{X}_i \to B$ and $\mathcal{D}_i$ be models two models of X_F and D respectively ($\mathcal{D}_i$ are divisors in $\mathcal{X}_i$). Then there is a constant C for which the following holds: Let $P \in X_F(\overline{F})$ be an algebraic point such that $P \notin D$. Let $h : B_P \to B$ be the corresponding curve and $P_i : B_P \to \mathcal{X}_i$ be the models of the point. Then*

$$\frac{1}{[F(P):F]} \cdot \left| N_{\mathcal{D}_1}^{(1)}(P_1) - N_{\mathcal{D}_2}^{(1)}(P_2) \right| \leq C.$$

Proof: The proof is similar to 4.11. We may suppose that there is a dominant map $g : \mathcal{X}_1 \to \mathcal{X}_2$. The divisor $g^*(\mathcal{D}_2) - \mathcal{D}_1$ is vertical. Thus there is a positive constant C and an effective divisor A on B such that $-Cf_1^*(A) \leq g^*(\mathcal{D}_1) - \mathcal{D}_2 \leq Cf_1^*(A)$. Thus, since $N_{\mathcal{D}_2}^{(1)}(P_2) = N_{g^*(\mathcal{D}_2)}^{(1)}(P_1)$ and $N_A^{(1)}(h) \leq \deg(A)[F(P):F]$, the conclusion follows.

Consequently we may give the following definition: Let X_F be a smooth projective variety. Let $Div_{snc}(X_F)$ be the set of simple normal crossing divisors of X_F. For every $D \in Div_{snc}(X_F)$, choose a model $h : \mathcal{X} \to B$ of X_F and $\mathcal{D}$ of D such that $\mathcal{D}$ is snc.

6.4 Definition. *Let L/F be a finite extension and $p \in X_F \setminus D(L)$. Let $P : B_p \to \mathcal{X}$ be the extension of the point. The class of the function $\frac{N_{\mathcal{D}}^{(1)}(P)}{[F(p):F]}$ in $H((X_F \setminus D)(\overline{F}); \mathbb{Z})$ depends only on p, X_F and D; We call it the* truncated counting function of D.

We also need the definition of the ramification term:

6.5 Definition. *Let L/F be a finite extension and $B_L \to B$ be the associated covering of curves. We will denote by $\chi(p)$ the number $\frac{\chi(B_p)}{[L:F]}$.*

One observe that $\chi(p)$ measure the ramification of the covering $B_p \to B$. Indeed, if such a covering is non ramified, $\chi(p)$ is just $\chi(B)$.

6.6 Conjecture. *(Strong ABC conjecture) Let F be a function field and $f : X_F \to B$ be a smooth projective variety over F. Let D be a simple normal crossing divisor. Let*

$\epsilon > 0$. *Then there is a proper closed subset $Z \subsetneq X_F$ such that, for every $p \in (X_F \setminus D)(\overline{F})$ we have*

$$h_{K_{X_F}(D)}(p) \leq (1+\epsilon)(\chi(p) + N_D^{(1)}(p)) + O(1).$$

The involved implicit constant may depend on ϵ.

This conjecture is very deep and perhaps it is not true in such a generality. Perhaps the factor $(1+\epsilon)$ is too ambitious and a more prudent conjecture is obtained by replacing this factor by some bigger one. In any case, since at the moment we cannot prove or disprove it in general we leave here the most ambitious version of the conjecture.

6.7 *Consequences of the main conjecture and remarks.*

(a) Suppose that X_F is an general non isotrivial variety of general type, this means that K_{X_F} is big and X_F do not contain rational curves. Then, if the conjecture is true for X_F and $D = \emptyset$, for every finite extension L/F, the set $X_F(L)$ is *not* Zariski dense. To prove this, it suffices to apply the conjecture and proposition 4.28.

(b) Suppose that X_F is general, non isotrivial and D is a snc divisor on it such that $K_{X_F}(D)$ is big. In this case we will say that the quasiprojective variety $U := X_F \setminus D$ is a general variety of log–general type. Fix a model $\mathcal{X}$ of X_F and $\mathcal{D}$ of D without vertical components. If the conjecture is true, then for every finite extension L/F and finite set of points $S \subset B_L$, the subset of $(\mathcal{D}, S)$–integral points of $X_F(L)$ is not Zariski dense. Observe that, if X_F is isotrivial we can only conclude that the $(\mathcal{D}, S)$–integral points form a bounded family (even if $\mathcal{D}$ is not isotrivial).

(c) Suppose that X_F is a variety over F and D_X is a snc divisor on it for which the main conjecture holds. Then we have

6.7 Proposition. *Suppose that the conjecture holds for the couple (X_F, D). Let $f : Y_F \to X_F$ be a finite morphism whose branch locus is D. Suppose that Y_F is non isotrivial and of general type. Then for every finite extension L/F the set $Y_F(L)$ is not Zariski dense.*

6.8 Remark. This proposition is essentially due to Elkies in the one dimensional case.

Proof: Let $D_Y = f^*(D)$. Let $p \in Y_F(L)$, one easily sees that $N_{D_Y}^{(1)}(p) = N_D^{(1)}(f(p))$. Thus, since the conjecture holds for (X_F, D) we have that, outside the involved closed set $f^{-1}(Z)$,

$$h_{f^*(K_{X_F}(D)}(p) \leq (1+\epsilon)(\chi(p) + N_{D_Y}^{(1)}(p)) + O(1).$$

On the other direction, the Hurwitz formula gives

$$K_{Y_F}(D_Y) = K_{Y_F}(\log D_Y) = f^*(K_{X_F}(\log D)).$$

Thus, since $N_{D_Y}^{(1)}(p) \leq h_{D_Y}(p) + O(1)$, we obtain

$$h_{K_Y}(p) \leq (1+\epsilon)(\chi(p) + \epsilon h_{D_Y}(p)) + O(1).$$

Since, by hypothesis, K_{Y_F} is big we may find a divisor H on Y_F and a constant C such that, for every point $q \in (Y_F \setminus H)(\overline{F})$

$$h_{K_{Y_F}}(q) \geq C \cdot h_{D_Y}(q).$$

Consequently, taking ϵ so small that $\epsilon' := \epsilon \cdot C \ll 1$, we get

$$(1-\epsilon')h_{K_Y}(p) \leq (1+\epsilon)\chi(p) + 0(1).$$

The conclusion follows.

6.9 Remark. Observe that, in order to have the previous consequence, the factor $(1+\epsilon)$ is essential. a different bigger factor is not enough.

As corollary of the proof we proved

6.10 Corollary. *Let $f : Y_F \to X_F$ be a finite morphism. Suppose that the branch locus D_X is a snc divisor. Let $D_Y = f^*(D_X)$. Then the conjecture holds for the couple (X_F, D_X) if and only if it holds for the couple (Y_F, D_Y).*

(d) Let L/F be a finite extension, X_L and D_L the base change of X_F and D respectively. Then the conjecture holds for the couple (X_F, D) if and only if it holds for (X_L, D_L). The proof of this is straightforward and left as exercise.

(e) (Very important complement to (d)) When (X_F, D) is a isotrivial curve with a isotrivial divisor, then the conjecture reduces to the Hurwitz formula:

6.11 Proposition. *Suppose that X_F is an isotrivial curve and D is an isotrivial divisor. Then, for every $p \in (X_F \setminus D)(\overline{F})$ we have that*

$$h_{K_{X_F}(D)}(p) = \chi(p) + N_D^{(1)}(p) + 0(1).$$

Thus, we see that in this one dimensional isotrivial case, the conjecture holds even with an equality and $\epsilon = 0$.

Proof: By (d) we may suppose that X_F and D are trivial. We may find a curve X_0 over k and a divisor D_0 over it such that $X_F = \simeq X_0 \times_k \mathrm{Spec}(F) \xrightarrow{h} X = 0$ and $D = h^*(D_0)$. Thus every point $p \in X_F(L)$ corresponds to a map $P : B_p \to X_0$ and $h_{K_{X_F}(D)}(p) = \deg(P^*(K_{X_0}(D)))$. Denote by D_P the divisor $P^{-1}(D_0)$ i.e. the unique reduced divisor and having the same support then $P^*(D_0)$. Observe that $\deg(D_P) = N_D^{(1)}(p)$. Hurwitz formula gives

$$P^*(K_{X_0}(D)) = K_{B_p}(D_P)$$

thus, taking degrees, the conclusion follows.

(e) Suppose that $X_F = \mathbb{P}^1_F$ and D is the the divisor $[0:1]+[1:0]+[-1:1]$. Then the conjecture becomes the theorem by Mason which is the analogous of the *abc*–conjecture over function fields: If B is a curve and $f \in k(B)$; let $(f)_0 = \sum_{i=0^N} n_i q_i$; we define $\deg(f) := \sum_i n_i$ and $N(f)$ to be the number $N := \sum \inf\{1, n_i\}$:

6.12 Proposition. *Let $f_1, f_2, f_3 \in k(B)$ rational functions such that $f_1 + f_2 = f_3$ then*

$$\max\{\deg(f_i)\} \leq \chi(B) + N(f_1) + N(f_2) + N(f_3) + C.$$

6.13 Exercise. Show that the statement above is equivalent (in this case) to the statement of conjecture 6.6.

6.14 Remark. This is the reason why conjecture 6.6 is called the *strong abc conjecture*.

6.16 *The weak version of the conjecture..*

We will state now a weaker version of the conjecture (but still very interesting and open) and prove it in an interesting case.

6.16 Conjecture. *(Weak ABC conjecture) Let F be a function field and $f : X_F \to B$ be a smooth projective variety over F. Let D be a simple normal crossing divisor. Then there is a constant A and a proper closed subset $Z \subsetneq X_F$ such that, for every $p \in (X_F \setminus D)(\overline{F})$ we have*

$$h_{K_{X_F}(D)}(p) \leq A(\chi(p) + N_D^{(1)}(p)) + O(1).$$

6.17 Remark. Observe that consequences (a) and (b) holds also in this case. Consequence (c) of the strong ABC conjecture is not consequence of the weaker version of it. Nevertheless a version of corollary 6.10 holds.

Let's restrict our attention to the case of curves. We will start with a very easy proof of the conjecture 6.16. In the proof of it we will introduce many of the tools which are used to prove stronger versions of the conjecture.

6.18 Theorem. *(Weak ABC for curves) Let X_F be a curve over F and D be a reduced divisor over it. Then there is a constant A such that, for every $p \in (X_F \setminus D)(\overline{F})$*

$$h_{K_{X_F}(D)}(p) \leq A(\chi(p) + N_D^{(1)}(p)) + O(1).$$

This theorem has as a corollary the Mordell conjecture over function fields:

6.19 Theorem. *Let X_F be a curve of genus at least two over the function field F. Suppose that X_F is not isotrivial, then, for every finite extension L/F, the set $X_F(L)$ is finite.*

It also implies an analogue of Siegel theorem on elliptic curve.

6.20 Theorem. *Let F be a function field and S be a finite set of points of B_F. Let E_F be a non isotrivial curve of genus one over F and D a reduced effective divisor over it. Then for every model $\mathcal{E} \to B_F$ of E and $\mathcal{D}$ of D, the set of $(\mathcal{D}, S)$ points of $\mathcal{E}$ is finite.*

Theorem 6.18 is a particular case a more general statement, which is essentially due to Moriwaki [Mo1]. Before that we need a definition.

6.21 Definition. *Let X be a smooth projective variety and E be a vector bundle over it. We will say that E is an ample vector bundle if the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ over the variety $\mathbb{P}(E)$.*

We observe the following properties of ample vector bundles:
(a) If E is ample over X and $Y \subseteq X$ is a subvariety, then $E|_Y$ is ample over Y
(b) If $E \twoheadrightarrow Q$ and Q is a vector bundle, then Q is ample.
(c) if $h : Y \to X$ is a finite covering, then $h^*(E)$ is ample.

6.22 Exercise. Prove (a), (b) and (c) above.

The theorem we prove now is the following:

6.23 Theorem. *Let X_F smooth non isotrivial projective variety over F and D be a snc divisor over it. Suppose that $\Omega^1_{X_F/F}(\log(D))$ is ample, then there is a constant A and a proper closed subset $Z \subsetneq X_F$ such that for every $p \in (X_F \setminus Z)(\overline{F})$ we have that*

$$h_{K_X(D)}(p) \leq A(\chi(p) + N^{(1)}_D(p)) + O(1).$$

6.24 Corollary. *In the same hypothesis as theorem 6.23. If X_F is general, then for every finite extension L/F the set $X_F(L)$ is not Zariski dense.*

The proof of the corollary is a straightforward application of the height theory.

6.25 Remark. (a) Observe that if X_F is a variety with ample cotangent bundle, then by the properties of ample bundles, it cannot contain rational or elliptic curves.

(b) If X_F has ample cotangent bundle and $Y \hookrightarrow X_F$ is a closed subvariety, then Y has ample cotangent bundle. One can find examples of varieties with ample cotangent bundles in [Db].

Proof: (of Theorem 6.23) Fix a model $f : \mathcal{X} \to B$ of X_F and $\mathcal{D}$ of D such that $\mathcal{X}$ is smooth projective over k and $\mathcal{D}$ is snc. By construction we have a non trivial extension over X_F

$$0 \to f^*(\Omega^1_{F/k}) \longrightarrow \mathcal{E} \longrightarrow \Omega^1_{X/F}(\log(D)) \to 0.$$

The vector bundle $\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))$ is a model of $\mathcal{E}$ over $\mathcal{X}$ and $h : \mathbb{P}(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \to \mathcal{X}$ is a model of $h : \mathbb{P}(\mathcal{E}) \to X_F$.

Let L/F be a finite extension and $p \in (X_F \setminus D)(L)$. Let $P : B_L \to \mathcal{X}$ be the model of the point p. By the functorial properties of the logarithmic differentials, we obtain a map

$$\alpha : P^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \to \Omega^1_{B_L/k}(\log(P^*(\mathcal{D}))).$$

Thus, by the functorial properties of relative differentials, a commutative diagram

$$\begin{tikzcd} & \mathbb{P}(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \arrow[d, "h"] \\ B_L \arrow[ur, "\tilde{P}"] \arrow[r, "P"] & \mathcal{X} \end{tikzcd}$$

such that $h \circ \tilde{P} = P$.

The image of α is a sublinebundle $\mathcal{L}$ of $\Omega^1_{B_L/k}(\log(P^*(\mathcal{D})))$. Moreover $\tilde{P}*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{L}$ and we obtain the important inequality

$$\deg(P^*(\mathcal{O}_{\mathbb{P}}(1)) \leq \chi(B_L) + N_D^{(1)}(P).$$

The theorem will be consequence of the following general theorem:

6.26 Theorem. *Let X be a smooth projective variety and M be an ample vector bundle over it. Suppose that*

$$0 \to \mathcal{O}_X \longrightarrow E \longrightarrow M \to 0$$

is a non trivial extension and N is a line bundle over X. Let $h : \mathbb{P}(E) \to X$ be the projective bundle associated to E and $\mathcal{O}(1)$ the tautological line bundle over it. Let Bl_n be the base locus of $H^0(\mathbb{P}(E), h^*(N) \otimes \mathcal{O}(n))$. Then, for $n \gg 0$ the map $h|_{Bl_n} : Bl_n \to X$ is not dominant.

Let's show how theorem 6.26 implies 6.23. Consider the line bundle $N := h^*(K_{\mathcal{X}}(\mathcal{D}))$ on $\mathbb{P}(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})))$. It is a model of $h^*(K_{X_F}(D))$ over $\mathbb{P}(\mathcal{E})$. Denote by Bl_n the base locus of $H^0(\mathbb{P}(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))), \mathcal{O}(n) \otimes N^{-1})$. By theorem 6.26, the for $n \gg 0$, restriction to the generic fibre of $h(Bl_n)$ is a proper closed subset, denote it by Z. Suppose that $p \notin Z(L)$. Then $\tilde{P}(B_L) \nsubseteq Bl_n$. Consequently the degree of $\tilde{P}^*(\mathcal{O}(n) \otimes N^{-1})$ is bigger or equal then zero. The conclusion follows.

Proof: (Of Theorem 6.26) By hypothesis we have an inclusion $Z := \mathbb{P}(M) \hookrightarrow \mathbb{P}(E)$ and $\mathcal{O}_{\mathbb{P}(E)}(\mathbb{P}(M)) = \mathcal{O}(1).$, Consequently, the line bundle $\mathcal{O}(1)|_Z$ is ample. Thus, as soon as n is sufficiently big, the linear system $H^0(Z, \mathcal{O}(n) \otimes N)$ is without base points and without higher cohomology. The exact sequence

$$0 \to \mathcal{O}(n-1) \otimes N \longrightarrow \mathcal{O}(n) \otimes N \longrightarrow \mathcal{O}(n) \otimes N|_Y \to 0$$

give rise, for $n \gg 0$, to a surjection

$$\alpha : H^1(\mathbb{P}(E); \mathcal{O}(n-1) \otimes N) \longrightarrow H^1(\mathbb{P}(E); \mathcal{O}(n) \otimes N) \to 0.$$

Thus for all $n \gg 0$, the dimension $h^1(\mathbb{P}(E), \mathcal{O}(n) \otimes N)$ is constant and the map α is an isomorphism; in particular it is injective. Consequently the restriction morphism

$$H^o((\mathbb{P}(E); \mathcal{O}(n) \otimes N) \longrightarrow H^0(Z, \mathcal{O}(n) \otimes N)$$

must be surjective. This implies that the base locus Bl_n cannot intersect Z. A similar argument shows that if Y is a sufficiently ample divisor of X, the natural restriction map

$$H^1(X, M^\vee) \longrightarrow H^1(Y, M^\vee|_Y)$$

is injective. Thus, by induction on the dimension, If C is a general curve in X, the extension

$$0 \to \mathcal{O}_C \longrightarrow E|_C \longrightarrow M|_C \to 0$$

is not split. Suppose that $h|_{Bl_n} :\to X$ is dominant. we may find a very general curve $C \subset X$ and a curve $C' \subset Bl_n$ such that $C = h(C')$. Since $C' \cap Z = \emptyset$ we have that $(\mathcal{O}(1), C') = 0$. The conclusion will follows from the following lemma:

6.27 Lemma. *Suppose that, in the hypothesis of theorem 6.26, we have that X is a smooth projective curve, then for every curve $C' \subseteq \mathbb{P}(E)$ we have that $(\mathcal{O}(1), C') > 0$.*

Proof: Let $\iota : C' \hookrightarrow \mathbb{P}(E)$ be a curve. Then $g := h|_{C'} : C' \to C$ is a finite morphism (we omit the trivial case when C' is contained in a fibre). By the functorial properties of $\mathbb{P}(E)$, the inclusion ι corresponds to a line bundle L on C' with a surjection $\beta : g^*(E) \to \to L$; moreover $\deg(L) = (\mathcal{O}(1); C')$. The exact sequence of the hypothesis give rise to an inclusion $i : \mathcal{O}_C \to g^*(E)$. Thus either $\beta \circ i : \mathcal{O}_{C'} \to L$ is the zero map or it is injective. In the first case, we get a non zero map $h^*(M) \to L$,thus, since M is ample, by properties (b) and (c) of ample vector bundles, $\deg(L) > 0$. In the second case either $\beta \circ i$ is injective or it is an isomorphism. If it is not an isomorphism, again $\deg(L) > 0$; otherwise, it give rise to a splitting of the the pull back of the exact sequence to C'; but this is not possible by 3.30.

7 The Vojta approach and counterexamples in positive characteristic.

In this section we will show the approach essentially due to Vojta and Kim to the ABC conjecture for curves over function fields. Instead of the factor $(1+\epsilon)$ in front of the characteristic class of the point, one obtain the factor $(2+\epsilon)$ in characteristic zero and $2g-2$, where g is the genus of the curve, in arbitrary characteristic when the Kodaira Spencer class is not vanishing. We will also show, with an example, that if the curve dominates an isotrivial curve, then, in positive characteristic, the factor $2g-2$ is optimal (this example is due to Voloch).

Even if in all the notes we supposed that the ground field k has characteristic zero, here we will not suppose it (if not explicitly said). This is important because the example we will describe shows that a proof of the general ABC conjecture (if it is true) must use considerations which hold only in characteristic zero.

In this section k will be an algebraically closed field *of arbitrary characteristic.* F will be a field of transcendence degree one over k. We recall (cf. for instance [Ei Corollary A.1.7]) that there is an element $t \in F$ such that F is a separable finite extension of $k(t)$. Thus $\Omega^1_{F/k} = F \cdot dt$. We denote as usual with B the smooth projective curve over k having as field of functions F (we will in general use the same notations as before).

Most of the theory developed in the previous sections carry on in this case. The main difference is that one cannot prove the finiteness theorem 4.26. In particular the we can define a Kodaira Spencer class for varieties defined over F.

One also have to be careful because at the moment we are not able to prove the resolution of singularities for varieties of dimension bigger or equal then 3 in arbitrary characteristic thus the theory of models of the varieties is more delicate. Never the less observe that we can perform many arguments using the alteration theorem by deJong [dJ].

Let $f : X_F \to \mathrm{Spec}(F)$ be a smooth projective curve of genus g and D be an effective closed divisor over it. Associated to this there is an exact sequence

$$0 \to f^*(\Omega^1_{F/k}) \longrightarrow \Omega^1_{\mathcal{X}_F/k}(\log(D)) \longrightarrow K_{X_F/F}(D) \to 0. \tag{7.1.1}$$

We will denote by $KS_D(f) \in Ext^1(K_{X_F/F}(D), f^*(\Omega^1_{F/k}))$ the class of this extension; it generalize the Kodaira Spencer class, we already used it in §4.

Notice that we can also consider the exact sequence

$$0 \to F\cdot dt \to H^0(X_F, \Omega^1_{X_F/k}(\log(D)) \to H^0(X_F, K_{X_F/F}(D)) \xrightarrow{ks_D} F\cdot dt \otimes H^1(X_F, \mathcal{O}_{X_F})$$

directly deduced from the exact sequence 7.1.1. Of course $KS_D(f)$ vanishes if and only if ks_D vanishes. We will say that (X_F, D) have *strong variation* if the map ks_D is injective. Notice that, since $h^0(X_F; K_{X_F/F}(D)) = g + \deg(D) - 1$ and $h^1(X_F; \mathcal{O}X_F) = g$, strong variation can happen only if $\deg(D) \le 1$.

7.1 Theorem. *Let $\epsilon > 0$. Suppose that the Kodaira Spencer class $KS_D(f)$ do not vanishes. Then for every $p \in X_F(\overline{F}) \setminus D$ we have that*

$$h_{K_{X_F/F}(D)}(p) \le (2g - 2 + \deg(D) + \epsilon)(\chi(p) + N^{(1)}_D(p)) + O_\epsilon(1)$$

(The involved constant depends on ϵ, the choice of the height and $N^{(1)}_D(\cdot)$).

If, moreover, (X_F, D) has strong variation, then

$$h_{K_{X_F/F}(D)}(p) \le (2 + \epsilon)(\chi(p) + N^{(1)}_D(p)) + O_\epsilon(1).$$

7.2 Remark. One should notice that, in arbitrary characteristic, a point is a morphism $p : Spec(L) \to X_F$ such that, if $\mathcal{X} \to B$ is a projective model of X_F and $P : B_p \to \mathcal{X}$ is the model of it, then the natural map $P^*(\Omega^1_{\mathcal{X}/k}) \to \Omega^1_{B_p/k}$ is *not* identically zero.

Proof: Since the resolution of singularities for surfaces is available in any characteristic, we can take a model $f : \mathcal{X} \to B$ of X_F which is regular and a snc divisor $\mathcal{D}$ in it which is a model of D. We can take a suitable blow up of $\mathcal{X}$ in such a way that The exact sequence 7.1.1 extends to an exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \to 0$$

where A_1, A_2 and A_3 are locally free and they are models of $f^*(\Omega^1_{F/k})$, $\Omega^1_{X_F/k}(\log(D))$ and $K_{X_F/k}(D)$ respectively. Let $U \subseteq B$ be the open set where the restriction of the morphism $f : \mathcal{X} \to B$ and the projection $f|_{\mathcal{D}} : \mathcal{D}|_U \to U$ are smooth; we may suppose that the restriction of A_1 to $V := f^{-1}(U)$ is isomorphic to $f^*(\Omega^1_{B/k})|_V$, the restriction of A_2 to V is isomorphic to $\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))|_V$ and the restriction of A_3 to V is isomorphic to $K_{\mathcal{X}/B}(D)|_V$.

Let $p \in X(\overline{F})$ and L_p be its field of definition. The lemma below tells us that we can suppose that the extension L_p/F is separable.

7.3 Lemma. *Suppose that the extension L_p/F is inseparable, then*

$$h_{K_{X_F/F}(D)}(p) \leq \chi(p) + N^{(1)}_D(p) + O(1).$$

Proof: Let $P : B_p \to \mathcal{X}$ be the model of the point p. Let $h : B_p \to B$ be the projection. Since L_p/F is inseparable, the natural morphism $dh : h^*(\Omega^1_B) \to \Omega^1_{B_p}$ is zero.

As in the proof of 6.23, we have a map

$$\alpha : P^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \to \Omega^1_{B_L/k}(\log(P^*(\mathcal{D}))).$$

Since the morphism dh vanishes, the morphism α induces an inclusion $P^*(C) \hookrightarrow \Omega^1_{B_{L_p}/k}(\log(P^*(\mathcal{D})))$. Thus $\deg(P^*(C)) \leq \chi(B_p) + N^{(1)}_D(p)$. Since C is a model of $K_{X_F/F}(D)$ the conclusion follows.

Denote by $\mathcal{E}$ the vector bundle $\Omega^1_{X_F/k}(\log(D))$. Consider the ruled surface $p : \mathbb{P}(\mathcal{E}) \to X_F$. Let $\mathcal{O}(1)$ be the tautological bundle over it. The self intersection $(\mathcal{O}(1); \mathcal{O}(1))$ is the degree of $\mathcal{E}$ so it is $2g - 2 + \deg(D)$. If $\deg(E) \leq 0$ there is nothing to prove so we will suppose that it is positive.

The self intersection of the $\mathbb{Q}$–line bundle $\mathcal{O}(2+\epsilon) \otimes p^*(K_{X_F/F}(D)^{-1})$ is $(2+\epsilon)^2(2g - 2 + \deg(D)) - 2(2+\epsilon)(2g - 2 + \deg(D)) = \epsilon(2+\epsilon)(2g - 2 + \deg(D)) > 0$. Since $\mathbb{P}(\mathcal{E})$ is ruled, there is no H^2 in the cohomology of line bundles, consequently, for $n \gg 0$ the line bundle $(\mathcal{O}(2+\epsilon) \otimes p^*(K_{X_F/F}(D)^{-1}))^{\otimes n}$ has global sections. Denote by Y_n the base locus of $H^0(\mathbb{P}(\mathcal{E}); (\mathcal{O}(2+\epsilon) \otimes p^*(K_{X_F/F}(D)^{-1}))^{\otimes n})$ and by Y the intersection of the Y_n's. We just shown that Y is not $\mathbb{P}(\mathcal{E})$.

The ruled threefold $p : \mathbb{P}(A_2) \to \mathcal{X}$ is a model of $\mathbb{P}(E)$. Denote by $\mathcal{O}(1)$ the tautological line bundle over it. it is a model of the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$.

As in the proof of 6.23, for every point $p \in X_F(\overline{F}) \setminus D$, the natural map $\alpha : P^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \to \Omega^1_{B_L/k}(\log(P^*(\mathcal{D})))$ induces a commutative diagram

$$\begin{array}{ccc} & & \mathbb{P}(A_2) \\ & \tilde{P} \nearrow & \big\downarrow p \\ B_L & \xrightarrow{P} & \mathcal{X} \end{array}$$

such that $p \circ \tilde{P} = P$. Consequently, we can find a positive constant A such that

$$\deg(\tilde{P}^*(\mathcal{O}(1)) \leq \chi(B_p) + N_D^{(1)}(p) + A[L_p : F].$$

Denote by $\tilde{p} : \mathrm{Spec}(L_p) \to \mathbb{P}(\mathcal{E})$ the generic fibre of $\tilde{P}$. Suppose that $\tilde{p}$ *do not* belong to Y, then by property (h) of heights and the formula above we have that

$$h_{K_{X_F/F}(D)}(p) \leq (2+\epsilon)(\chi(p) + N_D^{(1)}(p)) + O_\epsilon(1).$$

Now we have to deal with the points p such that $\tilde{p} \in Y$.

From now on, we will denote by Z an irreducible component of the base locus Y.

– *First case.* The morphism $p|_Z : Z \to \mathcal{X}$ is not dominant: In this case the image is either a closed point, thus it does contain any $\overline{F}$ point or it is a finite union of curves. The height of these curves can be inclosed in the constant. Thus the proof is finished in this case.

–*Second case.* The morphism $p|_Z : Z \to \mathcal{X}$ is dominant and separable: Over Z there is a splitting

$$0 \to M \longrightarrow p^*(A_2) \longrightarrow N \to 0 \tag{7.5.1}$$

induced by the corresponding splitting on $\mathbb{P}(A_2)$.

Let $r : \tilde{Z} \to Z$ be a desingularization of the surface Z where the divisor $(p \circ r)^*(\mathcal{D})$ is snc. The surface $\tilde{Z}$ is equipped with a fibration $\pi : \tilde{Z} \to B$. Taking the Zariski decomposition of π we find a covering $t : B' \to B$ and a fibration with connected fibres $s : \tilde{Z} \to B'$ such that $\pi = t \circ s$. Suppose that $P : B_p \to \mathcal{X}$ is a point whose generic point of $\tilde{P}$ is in Z. Then, since B_p is smooth, the map $\tilde{P}$ lifts to a map $B_p \to \tilde{Z}$, which, by abuse of notation, we denote again by $\tilde{P}$. By, 7.3, we may suppose that the covering $t : B' \to B$ is separable; indeed, otherwise the morphism $B_p \to B$, which factorizes through B' is also inseparable and we may apply loc. cit. Consequently we may suppose that $B' = B$. Denote by h the map $p \circ r : \tilde{Z} \to \mathcal{X}$.

The splitting 7.5.1 induces a splitting of $\tilde{A}_2 := h^*(A_2)$:

$$0 \to \tilde{M} \longrightarrow \tilde{A}_2 \longrightarrow \tilde{N} \to 0. \tag{7.6.1}$$

Suppose that $p \in X(\overline{F})$ is a point such that $\tilde{P} \in Z$. Then, again as in the proof of 6.23, we have that $\deg(\tilde{P}^*(\tilde{N})) \leq \chi(B_p) + N_D^{(1)}(p) + O(1)$. Suppose that $2\deg(\tilde{N}_\eta) \geq$

$\deg(h^*(K_{X_F/F}(D)))$ then property (g) of heights implies that

$$h_{h_{h^*(K_{X_F/F}(D))}}(\cdot) \leq (2+\epsilon)h_{\tilde{N}_\eta}(\cdot) + O(1)$$

and the conclusion follows.

Thus we may suppose that

$$2\deg(\tilde{N}_\eta) < \deg(h^*(K_{X_F/F}(D))).$$

7.6 Proposition. *The restriction to the generic fibre of the splitting 7.6.1 is the pull back to $\tilde{Z}_\eta$ of a splitting*

$$\mathcal{S} : 0 \to M_\eta \longrightarrow \mathcal{E} \longrightarrow N_\eta \to 0 \tag{7.7.1}$$

on X_F.

We show now how the proposition above implies the theorem, in this case. taking a suitable blow up, we may suppose that the exact sequence 7.7.1 extends to an exact sequence

$$0 \to M \longrightarrow E \longrightarrow N \to 0$$

of vector bundles over $\mathcal{X}$. Observe that E is a model of $\mathcal{E}$. Following the diagrams one sees that if a point $p \in X_F(\overline{F})$ is such that $\tilde{P} \in Z$, then

$$\deg(P^*(N)) \leq \chi(B_p) + N_D^{(1)}(p).$$

The degree of the line bundle N_η is positive otherwise the exact sequence 7.1.1 would be split against the hypothesis: the composite of the inclusion of $f^*(\Omega^1_{F/k})$ in $\mathcal{E}$ and the projection of $\mathcal{E}$ to N_η would give a splitting. Consequently, again by property (g) of heights

$$h_{K_{X_F/F}(D)}(\cdot) \leq (2g - 2 + \deg D + \epsilon)h_{N_\eta}(\cdot) + O(1)$$

and the conclusion follows.

The Proposition 7.6 will be consequence of the two lemmas below:

7.8 Lemma. *Let X be a smooth projective curve. Let E be a vector bundle of rank two over X. Then If E is not direct sum of two line bundles A and B such that $A \otimes B^{-1}$ is torsion, there is at most one splitting*

$$0 \to A \longrightarrow E \longrightarrow B \to 0$$

with $A \otimes B^{-1}$ is of positive degree or torsion.

7.9 Lemma. *Let $h : Z \to X$ be a Galois morphism between curves and E be a vector bundle over X. Let $M \hookrightarrow h^*(M)$ a sub line bundle such that, for every $\sigma \in Gal(Z/X)$*

we have that $\sigma^*(M) = M \hookrightarrow h^*(E)$. Then there is a sub line bundle $M_X \hookrightarrow E$ such that $h^*(M_X) = M$.

Let's see how the lemmas imply proposition 7.6. The condition on the degree of $\tilde{N}_\eta$ implies that $\deg(\tilde{M}_\eta) - \deg(\tilde{N}_\eta)$ is positive. Thus $\tilde{M}_\eta$ is the line bundle of the unique splitting of $h^*(\mathcal{E})$ having the properties of lemma 7.8. Since $h : \tilde{Z}_\eta \to X_F$ is separable, there is a covering $g : \tilde{Z}' \to \tilde{Z}_\eta$ such that $h \circ g$ is Galois. We may apply then 7.8 and 7.9 to the morphism $h \circ g$ and conclude.

Proof: (of Lemma 7.8). Suppose that we have two splitting

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & A_1 & & & \\ & & & \downarrow{\scriptstyle i_1} & & & \\ 0 \to A_2 & & \xrightarrow{i_2} & E & \xrightarrow{j_2} & B_2 \to & 0 \\ & & & \downarrow{\scriptstyle j_1} & & & \\ & & & B_1 & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

with $A_i \otimes B_i^{-1}$ of positive degree or torsion. then the composite map $j_1 \circ i_2$ and $j_2 \circ i_1$ give rise to an effective divisor D such that $B_1 = A_2(D)$ and $B_2 = A_1(D)$. Thus $A_1 \otimes B_1^{-1} = (A_2 \otimes B_2^{-1})^{-1}(-2D)$. Then $A_i \otimes B_1^{-1}$ may be both positive or torsion if and only if $D = 0$, $A_1 = B_2$, $B_1 = A_2$ and E is direct sum of A_1 and B_1.

Proof: (*of Lemma 7.9*). It suffices to prove the following: let L/F be a Galois extension of fields. Let $K \hookrightarrow L \oplus L$ a subspace such that, for every $\sigma \in G := Gal(L/F)$ we have that $\sigma(K) = K$ then there is a subspace K_F of $F \oplus F$ such that $K = K_F \otimes L$. To prove this it suffices to show that there is an element of K which belongs to $F \oplus F$: Let $(a, b) \in K$ be a non zero element. By hypothesis, for every $\sigma \in G$ there is an element $f_\sigma \in L$ such that $(\sigma(a), \sigma(b)) = f_\sigma(a, b)$. A simple calculation shows that, for σ and $\tau \in G$ we have that $f_{\tau\sigma} = \tau(f_\sigma) f_\tau$. Consider the element $v := \sum_{\sigma \in G} f_\sigma(a, b) \in K$. For every $\tau \in G$, $\tau(v) = \sum_{\sigma \in G} \tau(f_\sigma) f_\tau(a, b)$. Thus, by the observation above, we have that $\tau(v) = \sum_{\sigma \in G} f_{\tau\sigma}(a, b) = v$. The conclusion follows.

– *Third case.* The morphism $p|_Z : Z \to \mathcal{X}$ is dominant but inseparable: Factor $p|_Z : Z \to \mathcal{X}$ as $Z \xrightarrow{i} Z' \xrightarrow{s} \mathcal{X}$, where s is separable and i purely inseparable.

The map $\Omega_{Z'/k}(\log(s^*(\mathcal{D}))) \to i_*(\Omega^1_{Z/k}(\log(p^*(\mathcal{D}))))$ is *not injective* because i is inseparable. Let L be a saturated line bundle containing the kernel. Consider the map $\tilde{p}' := i \circ \tilde{p} : B_p \to Z'$; we claim that the induced map between differentials $\tilde{p}'^*(\Omega 1_{Z'/k}(\log(s^*(\mathcal{D})))) \to \Omega^1_{B_p/k}(\log(p^*(\mathcal{D})))$ factorizes through the quotient $\mathcal{G} :=$

$\Omega_{Z'/k}(\log(s^*(\mathcal{D})))/L$. Indeed, by functoriality we get a commutative diagram

$$\begin{array}{ccccc} 0 \to i^*(L) \longrightarrow i^*(\Omega_{Z'/k}(\log(s^*(\mathcal{D})))) & \longrightarrow & i^*(i_*(\Omega^1_{Z/k}(\log(p^*(\mathcal{D}))))) \\ & \searrow & \downarrow \\ & & \Omega^1_{Z/k}(\log(p^*(\mathcal{D}))) \end{array}$$

and since $i \circ \tilde{p} = \tilde{p}'$ the claim follows.

Since $s : Z' \to \mathcal{X}$ is separable, generically $s^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \to \Omega^1_{Z'/k}(\log(s^*(\mathcal{D})))$ is an isomorphism and thus the splitting

$$0 \to L \longrightarrow \Omega^1_{Z'/k}(\log(s^*(\mathcal{D}))) \longrightarrow \mathcal{G} \to 0$$

induces a splitting on $s^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})))$ and the natural map $\tilde{p}'^*(s^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})))) \to \Omega^1_{B_p/k}(\log(p^*(\mathcal{D})))$ factorizes through the kernel of it. We are now in a situation which is identical to the one in we treated in the separable case. Thus the conclusion follows in the same way.

We now come to the proof of part (b).

Let $p \in X_F(\overline{F}) \setminus D$. We see from the proof of part (a) that, either we can find a constant A and a splitting

$$0 \to L \to \Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})) \longrightarrow G \to 0$$

independent on p, such that

$$h_{K_{X_F/F(D)}}(p) \leq (2+\epsilon)(\chi(p) + N^{(1)}_D(p)) + A$$

or the natural map

$$p^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \longrightarrow \Omega^1_{B_p/k}(\log(p^*(\mathcal{D})))$$

factorizes through $p^*(G)$. In the last case $\deg(p^*(G)) \leq (\chi(B_p) + N^{(1)}_D(p))$.

If $\deg(G_\eta) \geq \frac{1}{2} \cdot (2g-2+\deg(D))$ then again property (g) of heights allow to conclude that

$$h_{K_{X_F/F(D)}}(\cdot) \leq (2+\epsilon) h_G(\cdot)$$

thus we conclude.

If $\deg(G_\eta) < \frac{1}{2} \cdot (2g-2+\deg(D))$, then $deg(L\eta) \geq g$ (or $\deg(L_\eta) > 0$ if $g = 0$); consequently $h^0(X_F, L) > 0$. In this case we get a commutative diagram

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ H^0(X_F, L) & & & & \\ \downarrow a & & & & \\ H^0(X_F, (\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})))_\eta) & \xrightarrow{b} & H^0(X_F, K_{X_F/F}(D)) & \xrightarrow{ks_D} & H^1(X_F, \mathcal{O}_{X_F}). \end{array}$$

The composite map $b \circ a$ cannot be zero otherwise the inclusion $L \hookrightarrow (\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})))_\eta$ would factorize through $f^*(\Omega^1_{F/k})$ which has degree zero and this is impossible because $\deg(L_\eta)$ is positive. Consequently the map ks_D cannot be injective. The conclusion follows.

7.10 *Counterexamples in positive characteristic.*

We want to show here that, in positive characteristic, we cannot obtain better then what we obtained in theorem 7.1.

Let p be a prime number and $\mathbb{F}_q$ be a field with q elements where $q = p^n$. Let $X/_{\mathbb{F}_q}$ be a variety defined over it. We recall that the *Frobenius morphism* is the morphism $F_X : X \to X$ which is the identity on the topological space and $f \to f^q$ on functions. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and $F_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ the Frobenius morphism of it. Denote $\overline{X}$ the $\overline{\mathbb{F}}_q$ variety $X \otimes_{\mathbb{F}_q} \mathrm{Spec}(\overline{\mathbb{F}}_q)$. We may consider the commutative diagram

$$\begin{array}{ccc}
\overline{X}^{(1)} := \overline{X} \times_{\overline{\mathbb{F}}_q} \mathrm{Spec}(\overline{\mathbb{F}}_q) & \longrightarrow & \overline{X} \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\overline{\mathbb{F}}_q) & \xrightarrow{F_q} & \mathrm{Spec}(\overline{\mathbb{F}}_q).
\end{array}$$

The Frobenius morphism $F_{\overline{X}} : \overline{X} \to \overline{X}$ give rise to a $\overline{\mathbb{F}}_q$*–morphism* $F^g_{\overline{X}} : \overline{X} \to \overline{X}^{(1)}$, usually called *the geometric Frobenius.*

Since $\overline{X}$ comes from a variety defined over $\mathbb{F}_q$, then $\overline{X}^{(1)}$ is isomorphic to $\overline{X}$ thus the geometric Frobenius is a $\overline{\mathbb{F}}_q$–endomorphism $F_g : \overline{X} \to \overline{X}$ of $\overline{X}$. Remark that the geometric Frobenius is purely inseparable.

Suppose that X is a smooth projective curve defined over $\mathbb{F}_q$. The geometric Frobenius of $\overline{X}$ is a flat morphism of degree p. In particular, if L is a line bundle over $\overline{X}$, then $\deg(F_g^*(L)) = p \deg(L)$.

From now on, in this section k will be the algebraic closure of $\mathbb{F}_q$.

7.10 *First example.* Suppose that C and S are smooth projective curves defined over $\mathbb{F}_q$. We denote by the same symbol the base change of them to k. Denote by F the function field of C. Suppose that we have a separable morphism $g : S \to C$. Denote by d the degree of g. The graph of g is an $\overline{F}$ point p_0 of the curve S, seen as a trivial curve over F. The product $S \times C$ is a model of S over C.

Denote by $F_S : S \to S$ the geometric Frobenius of S and consider the points $p_n : S \to S \times C$ given by the morphism (F_S^n, g).

We see that:

(a) $h_{K_S/F}(p_n) = \frac{1}{d}\deg(p_n^*(K_S)) = \frac{q^n}{d}\chi(S)$;

(b) $\chi(B_{b_n}) = \chi(S)$.

Thus we constructed a sequence of points on a trivial curve whose height goes to infinity but the Euler characteristic class of them remains constant. Consequently, if the Kodaira Spencer class vanishes, even the analogue of the weak *abc* conjecture cannot hold.

7.11 *Second example.*

We will show an example of curve which is not isotrivial but over it the strong version of the ABC conjecture do not holds. This example is due to Voloch.

Fix two smooth projective curves C and S. We fix a totally ramified morphism $g : S \to C$ of degree $d < p$ (the fact that g is totally ramified is not necessary but it simplify the proof). Denote by Γ_g the graph of g in $S \times C$.

We will show that there is an étale covering $\alpha : C' \to C$ and a generically finite morphism of degree d

$$b : X \longrightarrow S \times C'$$

ramified only over $(Id \times \alpha)^*(\Gamma_g)$.

Denote F the function field of S and by $X_F \to \mathrm{Spec}(F)$ the curve deduced from X. The curve X_F is *not isotrivial.* Indeed, fix a generic point $x \in S$ the fibre X_x of X over it is a curve with a morphism $b_x : X_x \to C'$ of degree d whose branch locus is the set of points y such that $\alpha(y) = g(x)$. If the family were isotrivial, the family of morphisms b_x were an infinite set of distinct morphisms from a fixed curve to another. And this cannot exists. Remark that the morphisms would be distinct because the branch locus varies.

The cover α give rise to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & C' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S & \xrightarrow{g} & C. \end{array}$$

Since α is étale and g totally ramified, the morphism g' is of degree d. Denote by $F_T : T \to T$ (resp. F_S, resp. F_C)the Frobenius morphism of T (resp. of S, resp. of C). The morphism $p_n := (h, g' \circ F_T^n) : T \to S \times C'$ give rise to a cartesian commutative diagram

$$\begin{array}{ccc} Q_n & \xrightarrow{b_n} & T \\ {\scriptstyle \tilde{p}_n}\downarrow & & \downarrow{\scriptstyle p_n} \\ X & \xrightarrow{b} & C' \times S. \end{array}$$

Notice that $\tilde{p}_n$ is a sequence of points in $X_F(\overline{F})$. The morphism $h \circ b_n : Q_n \to S$ is separable. The branch locus of it is contained in the set of $x \in S$ for which there exists $y \in S$ such that $g(x) = g(y)$ and $F_S^n(x) = F_S^n(y)$. since $F_S(g(x)) = g(F_C(x))$, we obtain that such a branch locus is contained in the set $g^{-1}(C(\mathbb{F}_{q^n}))$. By the Weil estimate on the number of rational points of a curve over a finite field, we have that $Card(C(\mathbb{F}_{q^n})) \sim q^n$. Consequently, the cardinality of the branch locus of $h \circ b_n$ is upper bounded by dq^n (plus error terms). From this we deduce that

$$\chi(Q_n) \leq A + \deg(b_n) \cdot \deg(h) \cdot q^n \cdot d$$

On the other side, the property (g) of heights tells us that there is a constant B such that

$$d \cdot \chi(C') \deg(\tilde{p}_n^*(K_{X/S})) = \chi(X_F) \cdot \deg((\tilde{p}_n \circ h)^*(K_{C' \times S/S})) + B.$$

Thus $\deg(\tilde{p}_n^*(K_{X/S})) \simeq \chi(X_F) q^n \deg(b_n)$. Consequently,

$$\frac{\deg(\tilde{p}_n^*(K_{X/S}))}{\chi(Q_n)} \geq \frac{\chi(X_F)}{d \deg(h)} - \epsilon$$

as soon as $n \gg 0$. Thus the strong ABC conjecture cannot hold for X.

We explain now how to construct such a X. This is a variation on a classical construction due to Kodaira. We keep the notations of the first example.

Let S be a smooth projective curve of genus bigger or equal then two. In 4.19 we introduced the Jacobian J(S) of S and the universal line bundle $\mathcal{P}^0$ over $S \times J(S)$. One can show that, for every integer n, the morphism multiplication by n, $[n] : J(S) \to J(S)$ is such that

$$(Id \times [n])^*(\mathcal{P}^0) \simeq (\mathcal{P}^0)^{\otimes n}. \qquad (7.11.1)$$

7.11 Lemma. *Suppose that the morphism g is of degree n and that $(n, p) = 1$. Denote by Γ_g the graph of g. There is an étale covering $\pi : C' \to C$ and a line bundle L on $C' \times S$ such that*

$$L^{\otimes n} \simeq (\pi \times Id)^*(\mathcal{O}(\Gamma_g)).$$

Proof: Fix a point $t \in S(k)$. The line bundle $L := \mathcal{O}(\Gamma_g - n\{t\}) \times C$ is of relative degree zero over C. By the universal property of the Jacobian, we can find a line bundle M on C and a morphism $f_L : C \to J(S)$ such that

$$(Id \times f_L)^*(\mathcal{P}^0) = L \otimes p_2^*(M).$$

Let $[n] : J(S) \to J[s]$ be the morphism multiplication by n. Let C' be the curve obtained by the cartesian diagram

$$\begin{array}{ccc} C' & \xrightarrow{f'} & J(S) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle [n]} \\ C & \xrightarrow{f_L} & J(S). \end{array}$$

Suppose for the moment that C is connected. The line bundle $r^*(M)$ is of degree divisible by n thus we can find a line bundle N' on C' such that $(N')^{\otimes n} \simeq r^*(N)$. By formula 7.11.1 we conclude in this case. In the general case we take an étale cyclic covering of a connected component of C' and apply the same argument.

The morphism $\pi : C' \to C$ give rise to an étale covering $\pi_S : C'_F \to C_F$. The morphism g corresponds to a point $p_0 \in C_F(F)$. We just proved that there exists a line bundle L on C'_F such that $\pi_S^*(\mathcal{O}(p_0)) \simeq L^{\otimes n}$. Let $s \in H^0(C_F, \mathcal{O}(p_0)))$ vanishing only on p_0. We may find a open covering $\{U_i\}$ of C'_F and a cocycle $\{g_{ij}\}$ for this covering which corresponds to the line bundle L. By definition we may suppose that s is given by local sections s_i such that $s_i = g_{ij}^n s_j$. Suppose that $U_i = \mathrm{Spec}(A_i)$ and consider the covering $V_i := \mathrm{Spec}(A_i[z_i]/(z_i^n - s_i) \to \mathrm{Spec}(A_i)$. We can glue together the V_i''s via the isomorphism $z_i \to g_{ij} z_j$ to obtain a covering $f : X_F \to S_{F'}$. Since the s_i vanish only on p_0, one easily (exercise: here we use that $d < p$) verify that the branch locus of f is p_0. The curve X_F is the searched curve.

7.14 *The Jouanoulou theorem.* In this subsection we suppose that k is a field of characteristic zero. In the proof of Theorem 7.1 we saw the following:

Suppose that $f : X_F \to \mathrm{Spec}(F)$ is a smooth projective curve and $D \subseteq X_F$ is a reduced effective divisor. Let $f : \mathcal{X} \to B$ and $\mathcal{D}$ be models of X_F and D respectively. Let $\epsilon > 0$. Then there is a constant A and finitely many rank one locally free sub bundles $L_i \hookrightarrow \Omega^1_{\mathcal{X}/k}$ such that the following happens:

Let $p \in X_F(\overline{F}) \setminus D$ then,

(a) Either $h_{K_{X_F/F}(D)}(p) \leq (2+\epsilon)(\chi(p) + N_D^{(1)}(p)) + A$;

(b) Or there is i such that the map $p^*(L_i) \to \Omega^1_{B_p/k}$, induced by the differential map $p^*(\Omega^1_{\mathcal{X}/k}) \to \Omega^1_{B_p/k}$ is the zero map.

The theorem by Jouanoulou is an interesting tool coming from the theory of foliations which tells us that, up to change the constant A, even in the situation (b) we may suppose that an inequality as in (a) holds.

7.14 Definition. *Let X be a surface over the field k. A* foliation over X is an exact sequence

$$\mathcal{F} : \quad 0 \to L \longrightarrow \Omega^1_{X/k} \longrightarrow G \to 0$$

where L is locally free and G is without torsion.

A foliation is essentially a differential equation on X.

7.15 Example. Let X be a smooth surface and B be a curve. Suppose that we have a flat morphism $\pi : X \to B$. The morphism π induces a foliation on X given by the exact sequence

$$0 \to L \longrightarrow \Omega^1_{X/k} \longrightarrow G \to 0.$$

where L the smallest subbundle of $\Omega^1_{X/k}$, containing $\pi^*(\Omega^1_{B/k})$ and such that $\Omega^1_{X/k}/L$ is without torsion. Observe that, due to the possibility to have multiple fibres, it may happen that $\Omega^1_{X/k}/\pi^*(\Omega^1_{B/k})$ has torsion. A foliation constructed in this way is called *a fibration.* More generally, we will say that a foliation is a fibration, if there is a blow up of X such that, the foliation induced on it is a fibration.

7.16 Definition. *Let*

$$\mathcal{F}: \quad 0 \to L \xrightarrow{\iota} \Omega^1_{X/k} \longrightarrow G \to 0$$

be a foliation on a surface X. Let C be a curve and $h : C \to X$ be an algebraic morphism. We will say that C, or more precisely h is tangent to $\mathcal{F}$ *if the composite map*

$$h^*(L) \xrightarrow{\iota} p^*(\Omega^1_{X/k}) \longrightarrow \Omega^1_{C/k}$$

is the zero map. In this case we say that C is a compact leaf *of the foliation.*

Observe that situation (b) tells us that there are finitely many foliations on $\mathcal{X}$ such that a point $p : B_p \to \mathcal{X}$, either satisfy the inequality in (a) or p is tangent to one of these foliations.

7.17 Remark. More generally, given a foliation, and a morphism of a disk $D \to X$, we may say that the disk is tangent to the foliation: the definition is given in this case as in definition 7.16 *mutatis mutandis*. We know, by the theorem of the existence and unicity of the solution of differentials equations, that, if the quotient sheaf is locally free around a point $p \in X$, then there is a unique disk D and a morphism $h : D \to X$ such that $h(0) = p$ and the image of D is tangent to the foliation. The image of h may be non algebraic, in particular Zariski dense.

The Jouanoulou theorem tells us that, given a foliation on a surface, then either there are only finitely many compact leaves or it is a fibration.

7.18 Theorem. *Let X be a smooth projective surface and $\mathcal{F}$ be a foliation over it. Suppose that $\mathcal{F}$ has infinitely many compact leaves, then $\mathcal{F}$ is a fibration.*

The theorem above has the following important consequence *in characteristic zero*:

7.19 Theorem. *Let X_F be a smooth projective curve and D be a reduced effective divisor over it. Let $\epsilon > 0$. Then there is a constant A_ϵ such that, for every $p \in (X_F \setminus D)(\overline{F})$ we have that*

$$h_{X_F/F(D)}(p) \leq (2+\epsilon)(\chi(p) + N^{(1)}_D(p)) + A_\epsilon.$$

Because of points (a) and (b) above, to prove it we have to deal only with points whose models are leaves of a given foliation. In this case, either they are only finitely many, so we may suppose that their height is less then the involved constant A_ϵ or they are infinitely many. In this case we apply theorem 7.18 and we deduce that there is a smooth projective curve C and a map $g : X \to C$ such that the model of each of these points are contained in the fibres of h. By Zariski decomposition theorem, we may suppose that the fibres of h are connected. But since the fibres of a morphism from a variety to a curve are numerically equivalent, the heights of these points is uniformly upper bounded. The conclusion follows.

Proof: (*of Theorem 7.18*) We start with a local remark: Let D_2 be a two dimensional ball, D_1 be a one dimensional ball and $f : D_2 \to D_1$ be an analytic map. Let $\omega \in \Omega^1_{D_2}$; it defines a foliation on D_2. Then each fibre of the map f is a leaf of the foliation if and only if $d(f)\wedge\omega = 0$. More precisely, ω will vanish along the fibres of f if and only if there are an holomorphic function h and a holomorphic 1 form α such that $\omega = h\cdot d(f)+f\cdot\alpha$. From this we deduce that each fibre of f is a leaf of the foliation defined by ω if and only if $\omega \wedge \frac{df}{f}$ is holomorphic.

Suppose that

$$\mathcal{F}:\ \ 0 \to L \longrightarrow \Omega^1_{X/k} \longrightarrow G \to 0$$

is the given foliation. It defines a global section $\omega \in H^0(X, \Omega^1_{X/k} \otimes L^{\otimes -1})$. By the remark above, we will conclude the proof if we show that the hypotheses imply that there is a non constant meromorphic function f such that $d(f) \wedge \omega = 0$.

Suppose that D is an effective divisor on X; suppose that over the open set U the divisor D is defined by the equation $f = 0$. The same argument given above tells us that D is a leaf of the foliation $\mathcal{F}$ if and only if $\frac{df}{f} \wedge \omega \in H^0(U, K_X \otimes L^{-1})$. Consequently, if D is snc, we have a natural map

$$\cdot \wedge \omega : \Omega^1_{X/k}(\log(D)) \longrightarrow K_X \otimes L^{-1}.$$

Let $D_1, \ldots, D_n$ be effective divisors on X which are compact leaves of $\mathcal{F}$. Taking a suitable blow up of X we may suppose that the divisor $D := \sum D_i$ is snc. Consider the map $\cdot \wedge \omega$ defined above and denote by $\mathcal{N}_n$ the kernel of it. Observe that $\mathcal{N}_n$ is of generic rank one.

A local computation (left as exercise) shows that, if $\tilde{X} \to X$ is a blow up, $\tilde{\mathcal{F}}$ the induced foliation on $\tilde{X}$, and $\tilde{L}$ the involved sub line bundle of $\Omega^1_{\tilde{X}}$, then $h^0(\tilde{X}; K_{\tilde{X}} \otimes \tilde{L}^{-1}) \leq h^0(X, K_X \otimes L^{-1})$. Consequently, if n is sufficiently big, then $h^0(X, \mathcal{N}_n) \geq 2$. Thus we may find two $\mathbb{C}$–linearly independent logarithmic differential forms α_1 and α_2 such that:

– $\alpha_i \wedge \omega = 0$;

– there is a meromorphic function h such that $\alpha_1 = h \cdot \alpha_2$. Moreover, h is not constant.

The lemma 7.20 below assures us that α_i are closed. Consequently

$$0 = d(\alpha_1) = d(h) \wedge \alpha_2 - h \cdot d(\alpha_2) = d(h) \wedge \alpha_2.$$

From this we conclude that h is a non constant meromorphic function on X such that $d(h) \wedge \omega = 0$. Thus the conclusion follows.

7.20 Lemma. *Let α be a global logarithmic form on a compact complex surface. Then $d(\alpha) = 0$.*

Sketch of Proof: The forms α and $d(\alpha)$ are integrable on X because they have only logarithmic poles. Consequetly, by Stokes Theorem

$$\int_X d(\alpha) = \int_{\partial X} \alpha.$$

Since X is without border, the last integral vanishes, consequently $d(\alpha) = 0$.

8 the proof of the $1 + \epsilon$ conjecture by McQuillan.

In this section we will give some details of the proof by McQuillan of the *abc* conjecture 6.6 in the case of curves. The proof uses tools from analytic geometry and the so called Semistable reduction theorem.

Let's recall the theorem we would like to prove:

8.1 Theorem. *Let F be a function field, X_F be a smooth projective curve over it and D be a reduced divisor on X_F. Let $\epsilon > 0$. Then, for every $p \in X_F(\overline{F})$ we have that*

$$h_{K_{X_F/F}(D)}(p) \leq (1 + \epsilon)(\chi(p) + N_D^{(1)}(p)) + O_\epsilon(1).$$

The involved constant depending only on ϵ and on the couple (X_F, D).

First of all we need to recall the semistable reduction theorem for curves. Let F be a function field and X_F be a curve over it. We saw that it is possible to find a surface $\mathcal{X} \to B$ with is is a model of X_F. The model is not unique and, even if it is a smooth surface, the morphism $f : \mathcal{X} \to B$ may be not very nice: fibers of it may be singular curves, or even non reduced. The semistable reduction theorem tells us that, making a finite extension of B if necessary, we can find a model such that the fibers of f are as nice as possible (in general we cannot hope that they are all smooth).

We need a definition:

8.2 Definition. *Let F be a function field, X_F be a smooth projective curve over it and D be an effective reduced divisor over it. A model $(\mathcal{X}, \mathcal{D})$ of the couple (X_F, D) of the couple (X_F, D) is said to be* semistable *if, denoting by $f : \mathcal{X} \to B$ the structural morphism:*

– $\mathcal{X}$ is a smooth projective surface and the fibres of f are connected;

– Let $S \subset B$ be the divisor where f is not smooth; then the divisor $\mathcal{D} + f^*(S)$ is a simple normal crossing divisor;

– if C is a rational curve contained in a fibre of f such that $(C; \mathcal{D}) = 0$ then $C^2 \neq -1$.

8.3 Remark. (a) Notice that in particular every fibre of f is a reduced simple normal crossing divisor;

(b) the third condition means that we cannot contract a rational curve contained in the fibre keeping $\mathcal{X}$ smooth.

The semistable reduction theorem is the following (and we admit it in this section)

8.4 Theorem. *Let X_F be a smooth projective curve over F and D be a reduced effective divisor over it. Then*

– If the couple (X_F, D) admits a semistable model and L/F is a finite extension, then the couple $(X_F \otimes L, D \otimes L)$ over L admits a semistable model.

– There exists a finite extension F'/F such that the couple $(X_F \otimes F', D \otimes F')$ admits a semistable model over the curve $B_{F'}$.

The second tool we will use is specific of the analytic geometry: *the theory of currents.* We cannot develop here all the details of the theory but we refer to for instance to [De] (or any other book on analytic geometry).

Let's recall some of the properties of the currents that we need.

Let X be a smooth variety of dimension n defined over the complex numbers.

– A (p,q)–form ω on X (or on a open set of it) is something which locally may be written as

$$\omega = \sum_{|I|=p, |J|=q} f_{IJ} dz_I \wedge d\overline{z}_J$$

where

(i) $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multi indices,

(ii) $|I| := p$, $|J| := q$,

(iii) dz_I means $dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\overline{z}_J$ means $d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$, where $z_1, \ldots, z_n$ are local holomorphic coordinates,

(iv) f_{IJ} are local C^∞ functions.

We will denote by $A^{(p,q)}(X)$ the space of the (p,q) forms of X.

– A (p,q) *current* T is something which locally may be written as

$$T = \sum_{|I|=n-p, |J|=n-q} T_{IJ} dz_I \wedge d\overline{z}_J$$

where everything is as above up to the fact that the T_{IJ} are local *distributions.* We will denote by $D^{(p,q)}(X)$ the space of the currents of type (p,q) over X. We can see $D^{(p,q)}(X)$ as the topological dual of the space $A^{(p,q)}(X)$.

–If T is (p,q) current and ω is a (r,s) C^∞ form, we can define $T \wedge \omega$. It will be a $(p-r, q-s)$ current.

– if ω is a form of type (p,q) with compact support and T is a current of type (p,q) then $T(\omega) \in \mathbb{C}$. We will use the notation $T(\omega) := \int_X T \wedge \omega$.

– As in the case of forms and in the theory of distributions, we can differentiate currents: If T is a current of type (p,q) then ∂T is of type $(p-1,q)$ while $\overline{\partial} T$ is a current of type $(p, q-1)$. We will denote by d the operator $\partial + \overline{\partial}$ on currents. A evident version of the Liebnitz law holds : Be careful: one can multiply only forms and currents, and not two currents. We have that $d^2 = 0$.

– If ω is a form of type $(n-p, n-q)$ it naturally defines a current T_ω of type (p,q): $T_\omega(\alpha) = \int_X \omega \wedge \alpha$. A current of this kind is said to be *smooth.* Thus we have a natural inclusion $A^{(p,q)}(X) \hookrightarrow D^{(n-p,n-q)}(X)$.

– A current T is said to be closed if $dT = 0$. Observe that, by definition, if T is closed, then $(d\alpha) = 0$. A current of type $(1,1)$ is said to be positive if, for every positive $(1,1)$ form α with compact support we have that $\int_X T \wedge \alpha > 0$.

–If $f : X \to Y$ is an analytic morphism and T is a current of type (p,q) on X, then we can naturally define $f_*(T)$ as $f_*(\alpha) := T(f^*(\alpha))$. The current $f_*(T)$ is naturally a (p,q) form on Y.

– If $Z \subset X$ is an analytic subvariety, of dimension p, it naturally defines a current $[Z]$ of type (p,p); it is defined as $[Z](\alpha) := \int_Z \alpha$. If Z is compact then the current $[Z]$ is closed.

– Suppose that X is compact. If L is a line bundle on X, we define its first Chern class in the following way: fix a smooth metric $\|\cdot\|$ on L, let s be a local holomorphic section of L, then the Poincaré–Lelong formula holds:

$$dd^c \log \|s\|^2 = [div(s)] - c_1(L; \|\cdot\|)$$

where $c_1(L)$ is a $(1,1)$–form depending only on L and the chosen metric, in particular independent on s. If we choose another metric $\|\cdot\|_1$ on L then there is a C^∞ function f such that

$$c_1(L, \|\cdot\|) - c_1(L, \|\cdot\|_1) = dd^c(f).$$

Consequently, if T is a *closed* $(1,1)$ form on X we may define unambiguously $T(L)$ as $\int_X T \wedge c_1(L, \|\cdot\|)$ where $\|\cdot\|$ is an arbitrary smooth metric on L.

– Suppose that X is compact. A Kähler metric ω on X is a closed positive $(1,1)$ form. It easily defines a measure on X. The Fubini–Study form on the projective space is Kähler. Consequently every projective variety is equipped with such a Kähler form.

– Let X be a compact complex manifold. Fix a finite covering $\mathcal{U} = \{U_i\}$ by open sets of X each of which is biholomorphic to a open ball of the right dimension. For each U_i fix an biholomorphic isomorphism of it with the ball. Let α be a (p,q) form over X. We may define the L^p norm of α, by taking the sup of L^p norms of the restriction of it to each U_i. This is not an intrinsic definition, but the topology on $A^{(p,q)}(X)$ associated to it is intrinsic. In this way we may also define the space $\mathbb{L}_p^{(p,q)}$ of forms with coefficients in the spaces L^p (in particular with $p = \infty$). A current of type (p,q) defines an element of the topological dual of $\mathbb{L}_\infty^{(p,q)}$. We denote the topological dual (with the weak topology) of $\mathbb{L}_\infty^{(p,q)}$ by $(\mathbb{L}_\infty^{(p,q)})^*$

An important tool we will need is the following:

8.5 Theorem. *Let X be a compact complex manifold equipped with a Kähler form ω. Suppose that T_n is a sequence of closed positive currents of type $(1,1)$ on X. The*

T_n's define currents

$$R_n : \mathbb{L}_\infty^{(p,q)} \longrightarrow \mathbb{C}$$
$$\alpha \longrightarrow \frac{T_n(\alpha)}{T_n(\omega)}.$$

Then, for every sequence (Q_n) of closed positive currents of type $(1,1)$ currents such that, for every positive $(1,1)$ form alpha we have that $Q_n(\alpha) \leq R_n(\alpha)$, there exists a subsequence Q_{n_k} converging, in the weak topology, to a closed positive current.

Proof: The only point is to prove that we can extract a sequence from the Q_n's which is convergent. The limit will be closed because each of the Q_n is closed.

Since X is compact and ω is positive, for each $(1,1)$ form α there exists a constant A such that $A\omega \pm \alpha$ is positive. Moreover, since the property of being positive is open, there is a sufficiently small $\epsilon > 0$ such that, if $\|\beta - \alpha\| < \epsilon$ then $A\omega \pm \beta$is again positive. This implies that we may find a constant C such that the norm of Q_n in the weak topology is bounded by C. Consequently, by the Banach–Alaoglu theorem, the conclusion follows.

8.6 Remark. The Banach–Alaoglu theorem sais that the unit ball of $(\mathbb{L}_\infty^{(p,q)})^*$ in the weak topology is compact.

– Suppose that X is as before. If $A \subseteq X$ is a subset, we will denote by $\mathbb{I}_A$ the characteristic function of A. Let E be an analytic subvariety of X. Let T be a closed positive current of type $(1,1)$. Consider a sequence of functions f_n which are smooth, $f_n \equiv 1$ in a neighborhood of E, with compact support and such that $\lim_{n\to\infty} f_n = \mathbb{I}_E$. The limit $\lim_{n\to\infty} f_n T$ is a closed positive current denoted by $\mathbb{I}_E T$. This is called the Skoda–El Mir Theorem. Denote by U the open set $X \setminus U$ then we denote by $\mathbb{I}_U T$ the current $T - \mathbb{I}_E U$. The current $\mathbb{I}_E T$ is again closed and positive.

– Suppose that T is closed positive of type $(1,1)$ over X. Let $\iota : E \hookrightarrow$ be a smooth divisor on X. Then there is a closed positive current S on E such that $\mathbb{I}_E T = \iota_*(S)$. For a proof of this cf. [Ga theorem 6.6].

8.7 *The proof of the strong ABC conjecture for curves.* We begin the proof of Theorem 8.1. First of all we remark that in order to prove the theorem we can make a suitable extension of F if necessary. Thus, by theorem 8.4, we may suppose that the couple (X_F, D) has a semistable model $f : \mathcal{X} \to B$ with divisor $\mathcal{D}$.Tensoring $K_{\mathcal{X}/k}(\mathcal{D})$ with the pullback of a line bundle on B with suitably big degree, we may suppose that the model $\mathbb{M}$ of $K_{X_F/F}(D)$ used to compute heights is nef and of the form $K_{\mathcal{X}/k}(\mathcal{D}) \otimes f^*(M)$. In particular we can fix a Kähler form ω on $\mathcal{X}$ and a metric on $\mathbb{M}$ in such a way that $c_1(\mathbb{M}) \geq \omega$ possibly only outside some vertical divisors.

8.8 *Construction of the currents.* Denote the projective bundle $Proj(\Omega^1_{\mathcal{X}/k}(\log \mathcal{D})) \to \mathcal{X}$ by $\pi : \mathbb{P} \to \mathcal{X}$, by $\mathbb{L}$ the tautological bundle over it and by $\mathbb{L}_1$ the line bundle $\mathbb{L} \otimes (f \circ \pi)^*(M)$.

Let Y be a smooth projective curve and $g : Y \to \mathcal{X}$ be a morphism. The couple (Y, g) defines a closed positive current T_Y of type $(1,1)$ over $\mathcal{X}$:

$$\begin{aligned} T_Y : & A^{(1,1)}(\mathcal{X}) \longrightarrow \mathbb{C} \\ & \alpha \longrightarrow T_Y(\alpha) := \int_Y g^*(\alpha). \end{aligned}$$

Remark that T_Y is closed by Stokes theorem and the projectivity of Y.

Let $\{(Y_n, g_n)\}$ be a sequence of curves and morphisms whose image is not contained in $\mathcal{D}$. By the remark above,

$$R_n := \frac{T_{Y_n}(\cdot)}{T_{Y_n}(\mathbb{M})} \leq \frac{T_{Y_n}(\cdot)}{T_{Y_n}(\omega)}.$$

Consequently, we may apply theorem 8.5 and deduce that we can find a subsequence R_{n_k} converging to a closed positive current $\mathbb{T}$ of type $(1,1)$ on $\mathcal{X}$.

Each of the g_n lifts to a morphism $g'_n : Y_n \to \mathbb{P}$ such that $\pi \circ g'_n = g_n$. Thus the sequence (Y_n, g'_n) define a sequence of closed positive currents T'_{Y_n} of type $(1,1)$ over $\mathbb{P}$.

In order to prove the theorem it suffices to prove the following theorem:

8.8 Theorem. *For every sequence (Y_n, g_n) as above and such that $f \circ g_n$ is dominant, we have that*

$$\liminf_n \frac{T'_{Y_n}(\mathbb{L}_1)}{T_{Y_n}(\mathbb{M})} \geq (1 - \epsilon).$$

8.9 Exercise. Prove that 8.8 implies 8.1.

In the sequel, we fix a sequence (Y_n, g_n) as in theorem 8.8 and such that

$$\liminf_n \frac{T'_{Y_n}(\mathbb{L}_1)}{T_{Y_n}(\mathbb{M})} = \lim_n \frac{T'_{Y_n}(\mathbb{L}_1)}{T_{Y_n}(\mathbb{M})} = a.$$

Of course, if $a = +\infty$ there is nothing to prove, thus we can suppose that $a < +\infty$. Theorem 8.8 is proved if we show that $a \geq 1 - \epsilon$. We may suppose that

$$\frac{T_{Y_n}(\mathbb{M})}{\deg(f \circ g_n)} \longrightarrow +\infty \tag{8.10.1}$$

otherwise there is nothing to prove. Observe that, if $p_n \in X_F(\overline{F})$ is the point corresponding to g_n, then $\frac{T_{Y_n}(\mathbb{M})}{\deg(f \circ g_n)} = h_{K_{X_F/F}(D)}(p_n)$.

We can find a positive constant A such that $\mathbb{L}_1 \otimes \pi^*(\mathbb{M}^{\otimes A})$ is nef on $\mathbb{P}$. Consequently, the same argument of the proof of 8.5 gives that the currents

$$\begin{aligned} R'_n : & A^{(1,1)}(\mathbb{P}) \longrightarrow \mathbb{C} \\ & \alpha \longrightarrow \frac{T'_{Y_n}(\alpha)}{T_{Y_n}(\mathbb{M})} \end{aligned}$$

are uniformly bounded. Thus we can find a subsequence of the R'_n converging to a closed positive current $\mathbb{T}'$ on of type $(1,1)$ on $\mathbb{P}$. By construction we have

$$\pi_*(\mathbb{T}') = \mathbb{T}.$$

The conclusion will follow if we show that

$$\mathbb{T}'(\mathbb{L} \otimes \pi^*(K_{\mathcal{X}/B}(\mathcal{D}))^{-1}) \geq -\epsilon. \tag{8.11.1}$$

Let $Z \subset \mathcal{X}$ be the set of points where the morphism f is not smooth. It is a finite set. Let I_Z be the ideal sheaf of Z.

8.13 *Restriction around the singularities.* Since $f : \mathcal{X} \to B$ is semistable, a local computation shows that the cokernel of the natural inclusion $f^*(\Omega^1_{B/k}) \hookrightarrow \Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))$ is $I_Z \otimes K_{\mathcal{X}/B}(D)$. Let $h : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up of $\mathcal{X}$ in Z and let E be the exceptional divisor on it. Thus we have an exact sequence

$$0 \to f^*(\Omega^1_{B/k}) \longrightarrow \Omega^1_{\mathcal{X}/k}(\log(\mathcal{D})) \xrightarrow{s} I_Z \otimes K_{\mathcal{X}/B}(\mathcal{D}) \to 0 \tag{8.13.1}$$

and the surjection s give rise to a surjection $h^*(\Omega^1_{\mathcal{X}/k}(\log(\mathcal{D}))) \twoheadrightarrow h^*(K_{\mathcal{X}/B}(D)(-E)$ thus to an inclusion

$$\iota : \tilde{\mathcal{X}} \longrightarrow \mathbb{P}. \tag{8.14.1}$$

By construction $\pi \circ \iota = h$ and $\iota^*(\mathbb{L}) = h^*(K_{\mathcal{X}/B}(D)(-E)$. Let $\Delta = \iota(\tilde{\mathcal{X}})$; it is a divisor in $\mathbb{P}$. Denote by U the open set $\mathbb{P} \setminus \Delta$.

A main step of the proof is the following important lemma:

8.15 Lemma. *We have that*

$$\pi_*(\mathbb{I}_U \mathbb{T}') = 0.$$

This lemma is crucial: we may think about it in the following way: Δ corresponds to the tangent vectors of $\mathcal{X}$ which are vertical with respect to f. We will see in the proof, that if a form α on $\mathcal{X}$ is such that $\pi^*(\alpha)$ has support in U, then α is dominated by a positive form coming from B. consequently, condition 8.10.1 will imply the lemma. Observe that we can also give an informal explanation of the lemma as follows: suppose that there is a sequence of curves that contradict the strong ABC conjecture, then these curves have to be not so ramified over B and with a very big area with respect to the relative dualizing sheaf. The only way to have that is that they become more and more vertical over B. Notice also that in this lemma we use the strength of the theory of currents.

Proof: We fix Kähler forms ω on $\mathcal{X}$ and η on B. In order to prove the proposition it suffices to prove the following: Let V be an open set of $\mathbb{P}$ such that $\overline{V} \cap \Delta = \emptyset$ ($\overline{V}$ being the closure of V in the Euclidean topology), then $\pi_*(\mathbb{I}_V T) = 0$. To prove this we claim that there exists a constant A_V (depending on V and the metrics) such that the following holds: if W is an open Riemann surface and $h : W \to X$ is an holomorphic map such that the image of $h' : W \to \mathbb{P}$ is contained in V, then $h^*(\omega) \leq A_V(f \circ h)^*(\eta)$. The claim implies the proposition. Indeed, let α be a form on X such that the support of $\pi^*(\alpha)$ is contained in U. Denote by Z the support of $\pi^*(\alpha)$; we may find a constant B such that $B\omega \pm \alpha$ is positive. Moreover we can find a smooth positive function φ on $\mathbb{P}$ which is always less or equal then one, it is identically one around Z and identically zero around Δ. This implies that $B\varphi\pi^*(\omega) \pm \pi^*(\alpha)$ is still positive. Let Y be a smooth projective curve and $g : Y \to X$ be a morphism such that $f \circ g$ is dominant, the image of which is not contained in $\mathcal{D}$. Let $g' : Y \to \mathbb{P}$ the induced map. By construction we have that $|T'_Y(\pi^*(\alpha))| \leq BT'_Y(\varphi\pi^*(\omega))$. By the claim, we can find constants A_i, depending only on the support of φ, such that

$$T'(\varphi\pi^*(\omega)) \leq A_1 \int_Y (f \circ g)^*(\eta) \leq A_2 \deg(f \circ g).$$

The conclusion follows because of property 8.10.1.

Fix such a V, Observe that $\pi(\overline{V})$ is a closed set of X which do not contains the singular points of the fibers P_i. By compactness of X, we can cover $\pi(\overline{V})$ by a finite set of disks B_j not containing the P_i's. We may restrict our attention to each of the B_j: thus we may suppose that:

– $\mathcal{X} = \{(z, w) \in \mathbb{C}^2 \ / \ |z| < 1 \ |w| < 1\}$, $B = \{z \in C \ /|z| < 1\}$ and $p(z, w) = z$;

– $\omega = \sqrt{-1}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ and $\eta = \sqrt{-1}(dz \wedge d\overline{z})$;

– $D = \{w = 0\}$ and the exact sequence 8.13.1 is the split exact sequence

$$0 \to \mathcal{O}_X dz \longrightarrow \mathcal{O}_X dz \oplus \mathcal{O}_X \frac{dw}{w} \longrightarrow \mathcal{O}_X \frac{dw}{w} \to 0;$$

– consequently $\mathbb{P} = X \times \mathbb{P}^1$ and $\Delta = X \times \{[0 : 1]\}$; we may then suppose that there exists a positive constant a such that $V \subseteq \{(z, w) \times [x : y] \ / \ |x|^2 > a|y|^2\}$.

– $W := \{z \ / \ |z| < 1\}$ and $h(z) = (h_1; h_2)$ and $h'(z) = (h_1; h_2) \times [h'_1 : \frac{h'_2}{h_2}]$.

The image of W via h' is contained in V, we have that $|h'_1(z)|^2 > a\left|\frac{h'_2}{h_2}\right|^2$. Thus $\frac{|h'_2(z)|^2}{|h'_1(z)|^2} < \frac{1}{a}$. Since $h(\omega) = \sqrt{-1}(|h'_1|^2 + |h'_2|^2)dz \wedge d\overline{z}$ and $h(\eta) = \sqrt{-1}(|h'_1|^2)dz \wedge d\overline{z}$ the proposition follows.

We can find an ample line bundle N on $\mathcal{X}$ such that $\mathbb{L} \otimes \pi^*(N)$ is ample on $\mathbb{P}$, consequently, $\mathbb{I}_U\mathbb{T}'(\mathbb{L} \otimes \pi^*(N) \geq 0$. By lemma 8.15, $\mathbb{I}_U\mathbb{T}'(\pi^*(N)) = \pi_*(\mathbb{I}_U\mathbb{T}'(N)) = 0$, thus $\mathbb{I}_U\mathbb{T}'(\mathbb{L}) \geq 0$.

On the other side, By the general properties of the closed positive currents, we may find a current S on Δ such that $\iota_*(S) = \mathbb{I}_\Delta\mathbb{T}'$. By construction, and again by lemma

8.15, $h_*(S) = \mathbb{T}$. Consequently

$$\begin{aligned}\mathbb{T}'(\mathbb{L} \otimes \pi^*(K_{\mathcal{X}/B}(\mathcal{D}))^{-1}) &= (\mathbb{I}_U + \mathbb{I}_\Delta)\mathbb{T}'(\mathbb{L} \otimes \pi^*(K_{\mathcal{X}/B}(\mathcal{D}))^{-1}) \\ &\geq \mathbb{I}_\Delta \mathbb{T}'(\mathbb{L} \otimes \pi^*(K_{\mathcal{X}/B}(\mathcal{D}))^{-1}) \\ &= S(\iota^*(\mathbb{L}) \otimes h^*(K_{\mathcal{X}/B}(\mathcal{D}))^{-1}) \\ &= S(-E).\end{aligned}$$

8.16 Remark. If f is smooth, then the proof is complete. The general case is more involved and as we can see the difficulties are localized around the singularities of f.

8.18 *Roots of the singular fibres.* In this subsection we will deal with the singular fibres of $f : \mathcal{X} \to B$. We should notice the following things:

– The problem is localized near the points of $\mathcal{X}$ where the projection f is singular.

– Here we will use in an essential way that the model $f : \mathcal{X} \to B$ is semistable.

This part also requires a delicate machinery: the theory of Deligne–Mumford stacks. The reader who do not like stacks may refer to [Ga] where a proof without stacks is given (nevertheless one should observe that on that proof one has to use singular varieties). The main idea is to take roots of the divisors which are the singular fibres.

In the sequel we will denote by $F = \sum F_i$ a singular fibre of f; observe that it is a s.n.c. divisor. Over F there are the points P where the morphism f is not smooth. For each of these points, denote by B and C the two components of the fibre which intersect in P.

The main tool we need is the following:

8.18 Theorem. *Let $m > 0$ be an integer. There is a Deligne–Mumford stack $p_m : \mathbb{X}_m \to \mathcal{X}$, smooth over $\mathbb{C}$ such that:*

– $\mathbb{X}_m$ is proper.

– Denote by $V \subset \mathcal{X}$ the open set $\mathcal{X} \setminus F$ then $p_m|_V : \mathbb{X}_m|_V \to V$ is an isomorphism.

– $p_m : \mathbb{X}_m \to \mathcal{X}$ is a finite morphism.

– $p_m^*(F_i) = m\tilde{F}_i$ and $\sum_i \tilde{F}_i$ is a simple normal crossing divisor on $\mathbb{X}_m$.

Before we start the proof we need the following construction: Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by multiplication. Then $\left[\mathring{A}^1/\mathbb{G}_m\right]$ the associated Artin Stack. A local computation shows that if X is a scheme (or more generally a stack), a morphism $X \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ is a couple (L, s) where L is a line bundle over X and $s \in H^0(X, L)$. In particular over $\left[\mathbb{A}^1/\mathbb{G}_m\right]$ we have the line bundle $\mathbb{A}^1 \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ with the section x (where $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$).

The raise to the m–th power on $\mathbb{A}^1$ and $\mathbb{G}_m$ induces a map $\varphi_m : \left[\mathbb{A}^1/\mathbb{G}_m\right] \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$. If X is a scheme and (L, s) is a map from it to $\left[\mathbb{A}^1/\mathbb{G}_m\right]$, then $\varphi_m \circ (L, s)$ is the map giving the couple $(L^{\otimes m}, s^{\otimes m})$ on X.

Similarly the group of the m–th roots of unity μ_m acts on $\mathbb{A}^1$ and $\left[\mathbb{A}^1/\mu_m\right]$ is a Deligne–Mumford stack (observe that we are using the fact that we are in characteristic zero). If Z is a scheme and$s \in \mathcal{O}_Z$ is a global section over it, we may consider the finite Z–scheme $Y_{s,m} := \mathrm{Spec}(\mathcal{O}_Z[z]/(z^m - s)) \to Z$. Over $Y_{s,m}$ there is a natural action of μ_m as follows: the action is trivial on $\mathcal{O}_Z$ and if ζ is a m–th root of unity, then $\zeta(z) = \zeta^{-1} \cdot z$. Let $[Y_{s,m}/\mu_m]$ be the associated Deligne Mumford Stack. A map from Z to $\left[\mathbb{A}^1/\mu_m\right]$ is the datum of a global section $s \in \mathcal{O}_Z$, and the stack $[Y_{s,m}/\mu_m]$.

We have natural maps $\iota : \left[\mathbb{A}^1/\mu_m\right] \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ and $j_m : \left[\mathbb{A}^1/\mu_m\right] \to \mathbb{A}^1$. From the description above we deduce that the following diagram is cartesian:

$$\begin{tikzcd} \left[\mathbb{A}^1/\mu_m\right] \arrow[r, "\iota"] \arrow[d, "j_m"'] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[d, "\varphi_m"] \\ \mathbb{A}^1 \arrow[r] & \left[\mathbb{A}^1/\mathbb{G}_m\right]. \end{tikzcd}$$

Denote by H_m the preimage of $\mathbb{G}_m$ via j_m. Observe that the map $j_m|_{H_m} \to \mathbb{G}_m$ is an isomorphism: usually one can informally describe this fenomenon by saying that the stacky structure of $[\mathbb{A}^1/\mu_m]$ is concentrated in 0.

8.19 Definition. *Let X be a scheme, L be a line bundle over it and $s \in H^0(X, L)$. By construction we have a map $k : X \to [\mathbb{A}^1/\mathbb{G}_m]$. Consider the cartesian diagram*

$$\begin{tikzcd} \mathbb{X}_m \arrow[r] \arrow[d, "k_m"'] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[d, "\varphi_m"] \\ X \arrow[r] & \left[\mathbb{A}^1/\mathbb{G}_m\right]. \end{tikzcd}$$

We will call the stack $k_m : \mathbb{X}_m \to X$ the stack of the m–th root of s.

By construction, a map from a scheme Z to $\mathbb{X}_m$ is a set (h, M, ι, t) where $h : Z \to X$ is a morphism, M is a line bundle over Z, $\iota : M \to h^*(L)$ is an isomorphism of line bundles and $t \in H^0(Z, M)$ is a global section such that $\iota(t^{\otimes m}) = h^*(s)$. Let $U := X \setminus \{s = 0\}$; then, as before, $k_m^{-1}(U) \xrightarrow{k_m} U$ is an isomorphism; thus the stacky structure of $\mathbb{X}_m$ is concentrated over $\{s = 0\}$.

8.20 Lemma. *$\mathbb{X}_m$ is a Deligne–Mumford stack.*

Proof: We know that it is an Artin Stack. Thus to verify the lemma we may work étale locally on X. Consequently we may suppose that L is trivial so the map $X \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ lifts to a map $X \to \mathbb{A}^1$. This implies that $\mathbb{X}_m = X \times_{\mathbb{A}^1} \left[\mathbb{A}^1/\mu_m\right]$. Since $\left[\mathbb{A}^1/\mu_m\right]$ is a Deligne–Mumford stack, the conclusion follows.

In general $\mathbb{X}_m$ may be singular, even not normal, but if X is smooth and $\{s = 0\}$ is a smooth divisor, then we have:

8.21 Lemma. *Suppose that X is a smooth variety and $\{s = 0\}$ is a smooth divisor, then $\mathbb{X}_m$ is smooth.*

Proof: We may work locally in the analytic topology. Let $D = \{x \in \mathbb{C} \ / \ |x| < 1\}$. Suppose that $X = D$ and $s = x$; in this case we see that $\mathbb{X}_n = [D/\mu_n]$ where if ζ is an m–root of unity then the action is $\zeta(x) := \zeta^{-}1 \cdot x$.

In general may suppose that $X = D^n$ with coordinates $x_1, \dots, x_n$ and $s = x_1$. Denote by $p : X \to D$ the projection on the first factor. We have have a cartesian diagram

$$\begin{array}{ccc} \mathbb{X}_m & \longrightarrow & [D/\mu_m] \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{p} & D. \end{array}$$

Since $[D/\mu_m]$ is smooth and the morphism p is smooth, the conclusion follows.

We can now prove theorem 8.18:

Proof: (*Of theorem 8.18)* Let $\{P_1, \dots, P_r\}$ be the set of points of $\mathcal{X}$ where f is not smooth and let $\mathcal{X}^0 = \mathcal{X} \setminus \{P_1, \dots, P_r\}$. The restriction F^0 of F to $\mathcal{X}^0$ is a smooth divisor.

Let $\mathbb{X}_m^0 \to \mathcal{X}^0$ be the stack of m–roots of F^0. By lemma 8.21, $\mathbb{X}_m$ is a smooth Deligne–Mumford stack.

We now study the local structure of $\mathbb{X}_m$ in the preimage of a neighborhood of the P_i's.

Let $Z \subset \mathbb{A}^3$ be the variety given by $z^m = xy$; μ_m acts over it by $\zeta(z) = \zeta^{-1} \cdot z$, $\zeta(x) = x$ and $\zeta(y) = y$. Let $\mathbb{B}_3$ be the unit ball in $\mathbb{C}^3$ and $\mathbb{B}_3^* := \mathbb{B}^3 \setminus \{(0,0,0)\}$. Denote by Z_1 the analytic variety $Z \cap \mathbb{B}_3$ and by Z_0 the analytic variety $Z \cap \mathbb{B}_3^*$. Eventually denote by $\tilde{Z}_0$ the smooth Deligne–Mumford stack $[Z/\mu_m]$.

Let D_2 be a open neighborhood of one of the P_i's which is biholomorphic to a 2–dimensional ball centered in P_i. We may suppose that z_1 and z_2 are coordinates on D_2 and that F is given by the equation $z_1 z_2 = 0$. Let $D_2^* := D_2 \setminus \{P\}$. The restriction of $\mathbb{X}_m^0$ to D_2^* is isomorphic to the stack $\tilde{Z}_0$. The stack $\tilde{Z}_0$ is an open set inside the stack $[Z_1/\mu_m]$ which is normal but not smooth.

We will now construct a smooth Deligne Mumford stack W with an open set W_0 such that:

– W_0 is isomorphic to $\tilde{Z}_0$;

– There is a finite map $W \to D_2$.

This is enough to conclude the proof: Indeed, in order to obtain $\mathbb{X}_m$, we glue the stack $\mathbb{X}_m^0$ with the stacks W's along the $\tilde{Z}_0$'s.

Let Δ be the 2 dimensional unit ball with coordinates ξ and η. Denote by Δ^* the surface $\Delta \setminus \{(0,0)\}$. Let G be the group $\mu_p \times \mu_p$ and H the subgroup $\mu_p \times \{1\}$. We let act G on Δ in the following way: $(\zeta_1, \zeta_2)(\xi) := \zeta_1 \cdot \zeta_2^{-1} \cdot \xi$, $(\zeta_1, \zeta_2)(\eta) := \zeta_1^{-1} \cdot \eta$. We have that $\Delta/H \simeq Z_1$ via the map $(\xi, \eta) \mapsto (\xi^m, \eta^m, \xi\eta)$. Since the action of H on Δ^* is free, we deduce that $[\Delta^*/H] \simeq Z_0$. The action of $\mu_m \simeq G/H$ on Z_0 coincide with the action of μ_m on it induced by the stack structure of $\mathbb{X}_m$. Consequently $[\Delta^*/G] \simeq [Z_0/\mu_m]$. Observe that, by construction, Δ is a smooth Deligne–Mumford stack.

The map $\Delta \to D_2$ given by $(\xi, \eta) \mapsto (\xi^m, \eta^m)$ give rise to a map $k : [\Delta/G] \to D_2$. The restriction of k to $[\Delta^*/G]$ coincide with the restriction of the projection $p_m : \mathbb{X}_m^0 \to \mathcal{X}^0$ to the preimage of D_2^*. Consequently we may put $W := [\Delta/G]$ and conclude.

The idea of the proof now is the following: We repeat the argument we made before with $\mathbb{X}_m$ replacing $\mathcal{X}_m$. Since $\tilde{F}_i = \frac{1}{m}p_m^*(F_i)$ we will see that the contribution given by the singularities will be smaller then before. Thus, taking m sufficiently big, the conclusion will follow.

Let $\mathcal{D}_m$ be the divisor $p_m^*(\mathcal{D})$ over $\mathbb{X}_m$. It is a snc divisor. Denote by $\Omega^1_{\mathbb{X}_m/k}(\log(\mathcal{D}_m))$ the sheaf of logarithmic differentials over $\mathbb{X}_m$ with poles along $\mathcal{D}_m$, by $\pi_m : \mathbb{P}_m \to \mathbb{X}_m$ the projective bundle $Proj(\Omega^1_{\mathbb{X}_m/k}(\log(\mathcal{D}_m))) \to \mathbb{X}_m$ and by $\mathbb{L}_m$ the tautological line bundle over it. We also fix a Kähler form ω_m on $\mathbb{X}_m$.

For every smooth projective curve Y with a morphism $g : Y \to \mathcal{X}$ such that $f \circ g : Y \to B$ is surjective denote by Y_m the normalization of a irreducible component of the stack $Y \times_{\mathcal{X}} \mathbb{X}_m$. Local computations give that:

– The projection $Y_{\mathfrak{m}} \to Y$ is finite.

– By the Hurwitz formula we obtain that

$$\chi(Y_m) \leq \chi(Y) + C \deg(f \circ g) \tag{8.22.1}$$

For a suitable constant C. Indeed the projection $r_m : Y_m \to Y$ is ramified at most over the points where the image of Y intersects the singular fibres F_i. Moreover, a local computation shows that over these points, the index of ramification is m. Since $g(Y)$ intersect the singular fibres in at most $\deg(f \circ g)$ points and over each of these points the local degree of the ramification point is $1 - \frac{1}{m}$ the conclusion follows.

– If N is a line bundle over Y, then $\deg(r_m^*(N)) = \deg(N)$.

The following definition is natural:

8.23 Definition. *Let $\mathbb{Y}$ be a smooth proper Deligne–Mumford of dimension one. For each closed point P on $\mathbb{Y}$ denote by G_P the stabilizer of P in $\mathbb{Y}$; notice that it is a finite group. Let $D\sum_i n_P P$ be an effective divisor on it, then we define $N^{(1)}_D(Y) := \sum_i \frac{1}{|G_P|} \min 1, n_P$.*

Of course the definition above naturally generalize to stacks the corresponding definition on smooth projective curves. Moreover we have:

– There is a constant C such that

$$N^{(1)}_{\mathcal{D}_m}(Y_m) \leq N^{(1)}_{\mathcal{D}}(Y) + \frac{1}{m}C.$$

From the sequence (Y_n, g_m) we can construct a sequence $(Y_{n,m}; g_{n,m})$.

We will denote by $f_m : \mathbb{X}_m \to B$ the map $f \circ k_m$. A local computation shows that

$$K_{\mathbb{X}_m/k}(\mathcal{D}_m) = p_m^*(K_{\mathcal{X}/k}(\mathcal{D}))((m-1)\sum \tilde{F}_i).$$

Consequently, the remarks above imply that if we prove that

$$\liminf_{n\to\infty} \frac{\chi(Y_{n,m}) + N^{(1)}(Y_{m,n})}{\deg(g^*_{m,n}(K_{\mathbb{X}_m/k}(\mathcal{D}_m)))} \geq 1 - \epsilon$$

then Theorem 8.1 follows.

As before, we can find a line bundle M_m on B such that $c_1(K_{\mathbb{X}_m/k}(\mathcal{D}_m) \otimes f_m^*(M_m)) \geq \alpha\omega_m$ and $c_1(p_m^*(K_{\mathcal{X}/k}(\mathcal{D}))) \geq \alpha\omega_m$ (for a suitable positive α) outside some vertical divisors. Denote by $\mathbb{M}_m$ the line bundle $K_{\mathbb{X}_m/k}(\mathcal{D}_m) \otimes f_m^*(M_m)$.

From the sequence sequence $(Y_{n,m}; g_{n,m})$ we may construct a sequence of closed positive currents

$$\begin{aligned} T_{Y_{n,m}} :& A^{(1,1)}(\mathbb{X}_m) \longrightarrow \mathbb{C} \\ \alpha \longrightarrow & T_{Y_{n,m}}(\alpha) := \frac{\int_{Y_{n,m}} g^*(\alpha)}{\deg(g_{n,m}(\mathbb{M}_m))}. \end{aligned}$$

As before, we may take a subsequence of the $Y_{n,m}$ and the sequence of the $T_{Y_{n,m}}$ will converge to a current $\mathbb{T}_m$ on $\mathbb{X}_m$ which is closed and positive of type $(1,1)$. Observe that $(p_m)_*(\mathbb{T}_m) = \mathbb{T}$.

The maps $g_{n,m}$ lift to maps $g'_{n,m} : Y_{n,m} \to \mathbb{P}_m$ and as before we can construct a closed positive current $\mathbb{T}'$ of type $(1,1)$ on $\mathbb{P}_m$ such that $(\pi_m) * (\mathbb{T}'_m) = \mathbb{T}'_m$.

As in the case of $\mathcal{X}$, the map f_m give rise to an exact sequence

$$0 \to A \longrightarrow \Omega^1_{\mathbb{X}_m/k}(\log(\mathcal{D}_m)) \xrightarrow{s_m} I_{Z_m} \otimes K_{\mathbb{X}_n/B}(\mathcal{D}_m) \to 0 \tag{8.24.1}$$

A is the saturated of the image of $\Omega^1_{B/k}$ inside $\Omega^1_{\mathbb{X}_m/k}(\log(\mathcal{D}_m))$ via f^* and Z_m is a zero cycle supported on the fibres over the P_i and locally given by the ideal (ξ^m, η^m). Denote by $\overline{Z}_m$ cycle which near Z_m is given by the ideal (ξ, η) (it is as if we take the reduced cycle associated to Z_m).

Let $h_m : \tilde{\mathbb{X}}_m \to \mathbb{X}_m$ be the blow up of $\mathbb{X}_m$ on $\overline{Z}_m$ and by E_n the exceptional divisor on it. The exact sequence 8.24.1 give rise to an inclusion

$$\iota_m : \tilde{\mathbb{X}}_m \longrightarrow \mathbb{P}_m$$

which is the analogue of 8.14.1. Of course, again we have $\pi_m \circ \iota_m = h_m$. Again to check this it suffices to work locally, thus we may suppose that $\mathbb{X}_m = D_2$ with coordinates (ξ, η) and the projection is $(\xi, \eta) \to (\xi\eta)^m$ (the details are left as exercise). Denote by Δ_m the image $\iota_m(\tilde{\mathbb{X}}_m)$ and by U_m the open set $\mathbb{P}_M \setminus \Delta_m$.

In the same way of 8.15 we can prove the following:

8.25 Lemma. *We have that*

$$(\pi_m)_*(\mathbb{I}_{U_m}\mathbb{T}'_m) = 0.$$

Moreover, there is a closed positive current S_m on $\tilde{\mathbb{X}}_m$ such that

$$(\iota_m)_*(S_m) = \mathbb{I}_{\Delta_m}\mathbb{T}'.$$

The conclusion will follow if we prove that $S_m(-E_m) \geq -\epsilon$.

Again a local computation shows that we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{X}}_m & \longrightarrow & \mathbb{X}_m \\ {\scriptstyle q_m}\Big\downarrow & & \Big\downarrow{\scriptstyle p_m} \\ \tilde{\mathcal{X}} & \xrightarrow{p} & \mathcal{X} \end{array}$$

with the following properties:

– $(q_m)_*(S_m) = S$;

– $q_m * (E) = mE_m$.

Indeed, if D_2 with coordinates (ξ, η) is a open set of $\mathbb{X}_m$ and D_2 with coordinates (z_1, z_2) is a open set of $\mathcal{X}$; then the map p_m is the map $p_m(\xi, \eta) = (\xi^m, \eta^m)$. The blow up of D_2 are locally given by $(\xi, v) \to (\xi, \xi v)$ on $\mathbb{X}_m$ and by $(z_1, u) \to (z_1, z_1 u)$ on $\mathcal{X}$ with exceptional divisors $\xi = 0$ and $z_1 = 0$ respectively. The commutative diagram and the properties follow.

The conclusion of the proof is now straightforward:

$$\begin{aligned} S_m(-E_m) &= S_m(-\frac{1}{m} q_m^*(E)) \\ &= -\frac{1}{m} \cdot (q_m)_*(S_m)(E) \\ &= -\frac{1}{m} \cdot S(E). \end{aligned}$$

Since we can take m as big as we want, the conclusion follows.

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